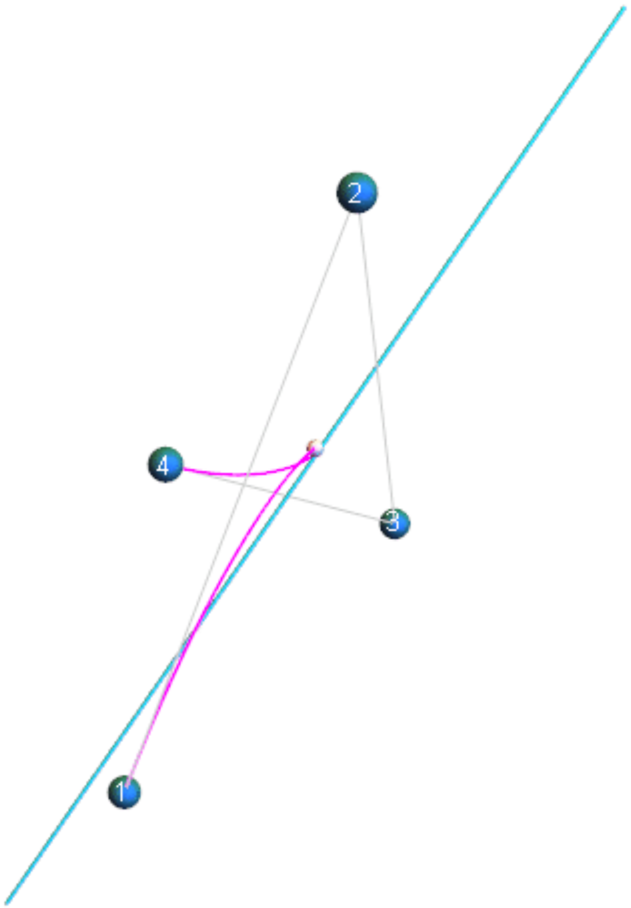
Exploiting Budan-Fourier and Vincent's Theorems for Ray Tracing 3D Bézier Curves

Alex Reshetov
NVIDIA



High-Performance Graphics 2017
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Why now

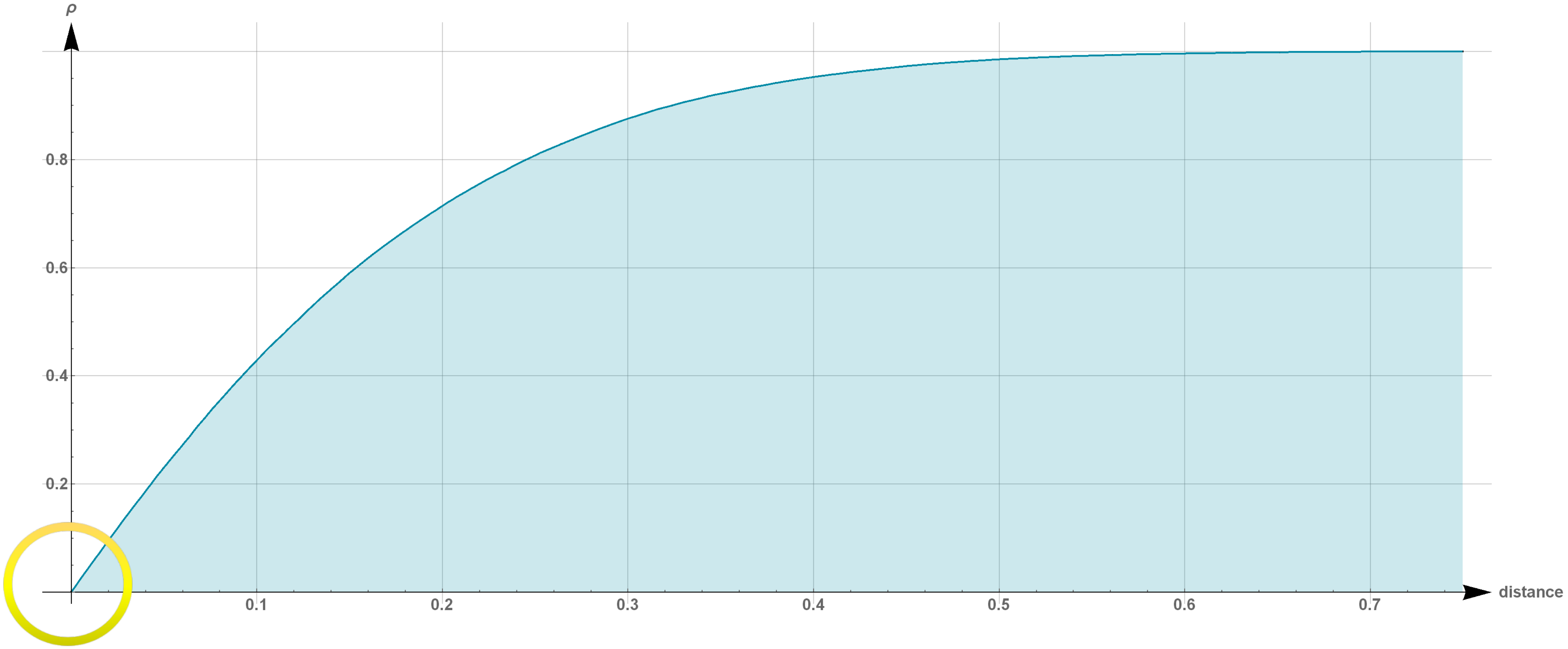
- $\exists$ need
  - Hair, grass, vegetation, contours, etc
  - Linear approximations or alpha blending are running its course
- Significant progress in studies of polynomials since Galois' last letter
  - Vincent–Akritas–Strzeboński (VAS, 2005) is used by Mathematica, Sage, etc
  - Math is mostly elementary
  - Far-reaching similarities between Bernstein polynomials and root-isolation techniques (not fully recognized yet)

What

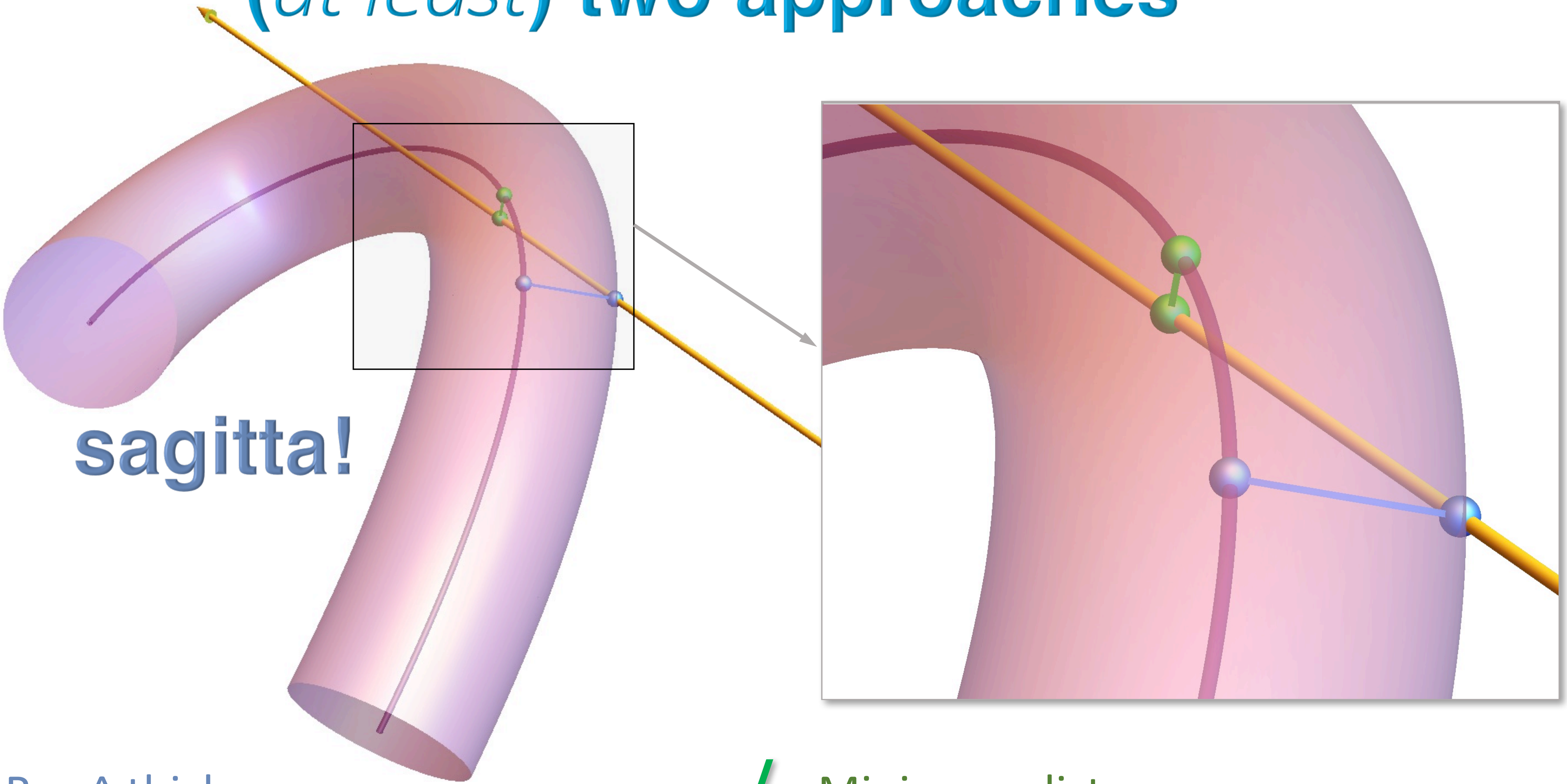
- Ray tracing cubic Bézier curves
  - i.e. finding curve's t for ray/curve intersection(s)
  - Rasterization is quite different ($t \rightarrow$ geometry)
  - Degree 3 is sufficient
    - It is expressive enough (inflection points & smooth connections)
    - Longer curves will be chopped into pieces anyway by acceleration structures
- Will *not* talk about
  - Building acceleration structures
  - Shading

What intersection?

probability of a ray to be closer to a curve than a given distance



(*at least*) two approaches



- Ray ^ thick curve
  more thrilling

√ Minimum distance
more universal (and easier)

Why *t*?

It looks simple…

Why *t*?

…but it isn’t

- $t \rightarrow$ geometry is easy
- geometry $\rightarrow t$ is not
- Yet we have to recover parameters
  to keep everything in an analytical form
  - normals
  - texture maps
  - etc

These approximate 2D methods will not work

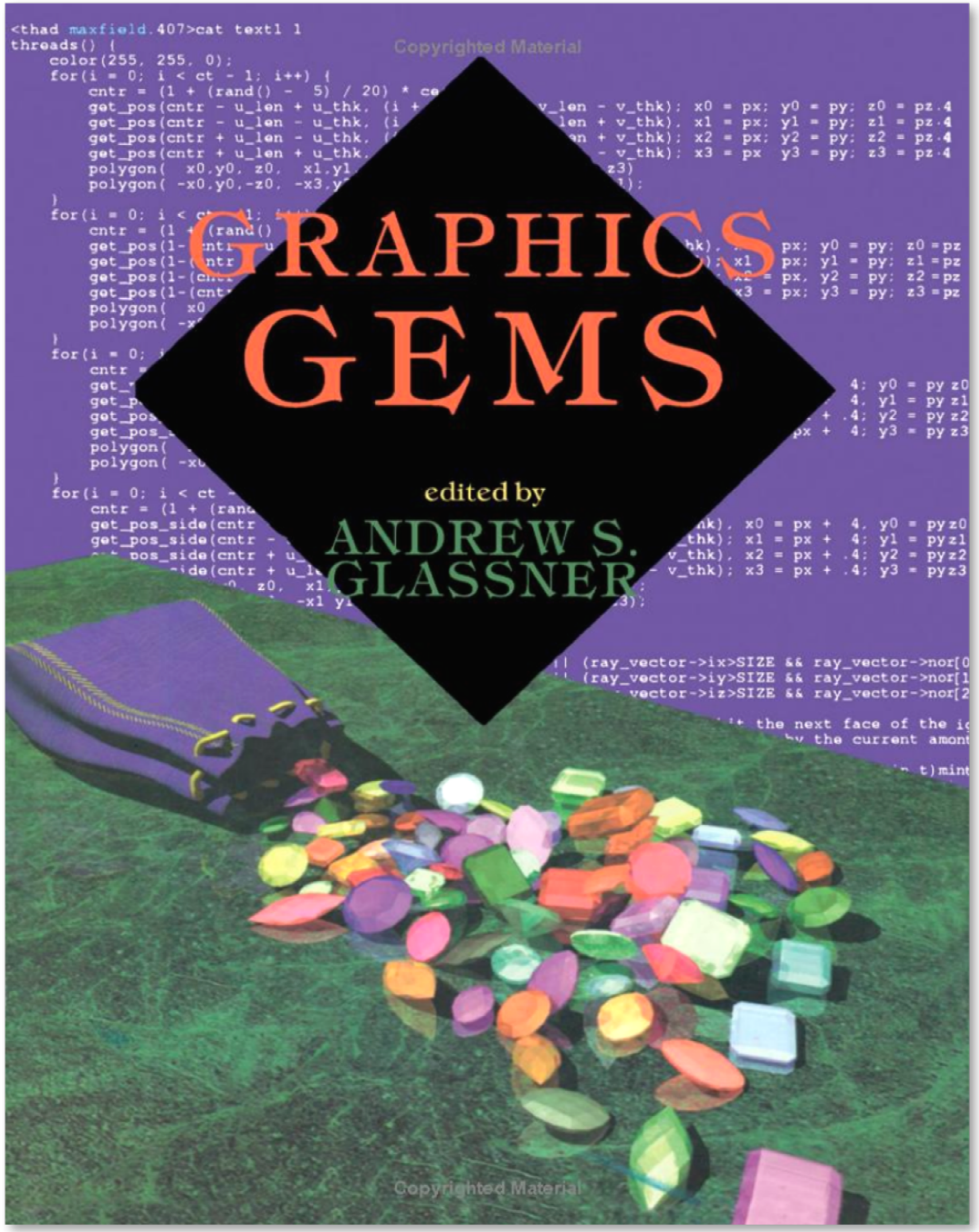
Batra et al. 2015;
Ganacim et al. 2014;
Kilgard and Bolz 2012;
Liao et al. 2012;
Loop and Blinn 2005;
Nehab and Hoppe 2008, 2012;
Qin et al. 2008;
Ray et al. 2005;
Reshetov and Luebke 2016;
Sen 2004;
Sun et al. 2012;
Tarini and Cignoni 2005

not because it is 2D,

these methods will not produce

a (good) curve's parameter ***t***

Project's inspiration



- many excellent articles on parametric representation of geometry
- Once you use a parametric form, everything is "equation solving" (either explicitly or implicitly)
- Hook and McAree [1990] introduced (to graphics community) Sturm sequences for solving equations

(my understanding of Sturm sequences)

DAVE DORMAN

State-of-the-art: adaptive linearization

Sederberg and Nishita [1990];

Nakamaru and Ohno [2002];

Barringer et al [2012];

Qin et al [2014];

Woop et al [2014];

Chiang at al [2015]

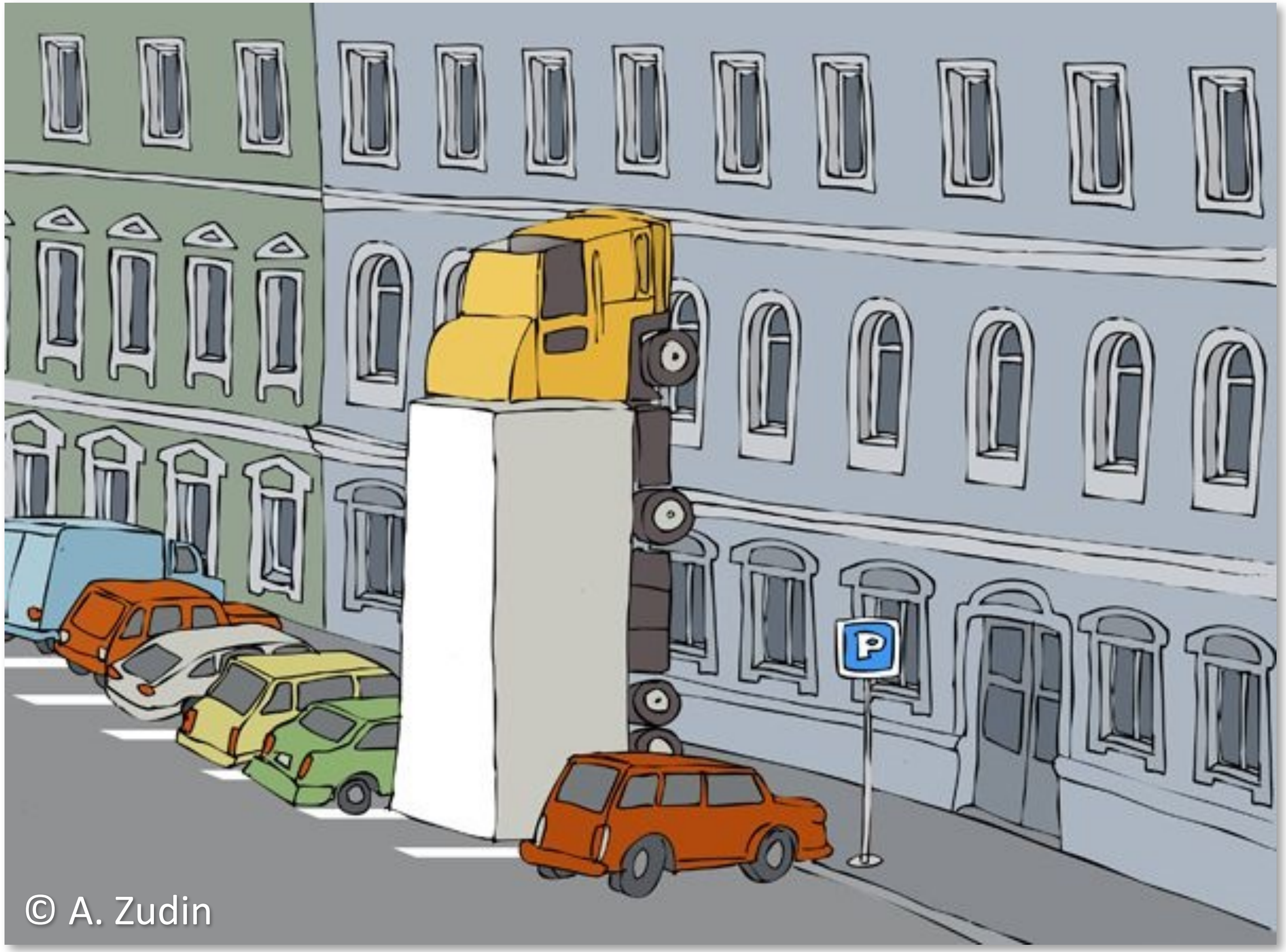
- Just chop the curve into pieces until it can be safely approximated with a line
- “Chopping” is done by (recursively) splitting the curve in half during rendering
- Works best for almost-linear curves; the curves are pre-split by an acceleration structure anyway

Why state-of-the-art is *sm*art

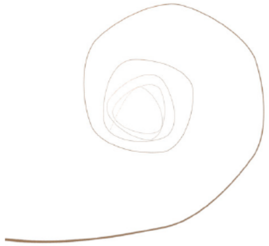
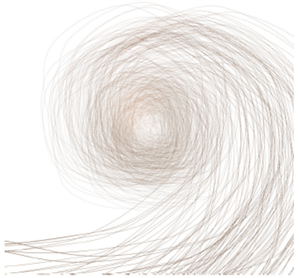
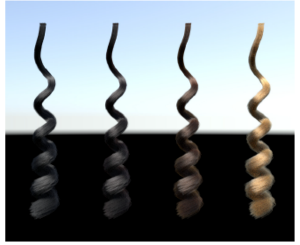
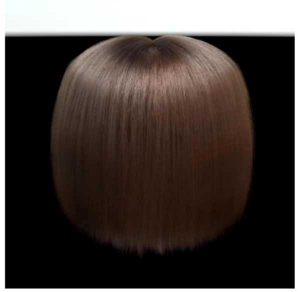
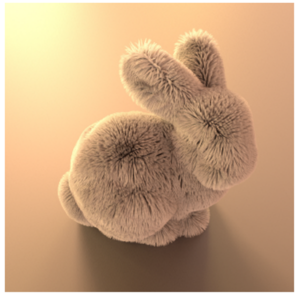
- It is effectively a bisection search of polynomial roots (corresponding to the minimum distance) while simultaneously
- isolating roots of the equation.
- Splitting in half can be done efficiently by using curve’s control points

- Alternatives (~Hook and McAree) are more challenging:
  - Computing Sturm sequences requires a long polynomial division
  - Traditional root-finding techniques (Newton-Raphson, secant, etc) have a lot of issues due to the target function non-linearity

Proposing something drastically different



Budan vs adaptive linearization

	 1. single hair	 2. lock	 3. curls	 4. straight	 5. curly	 6. fur
average kernel time in nanoseconds	13.6 675.6	33.1 675.2	57.0 697.0	116.0 793.4	145.6 1113.0	90.8 753.9
average distance error	7.7×10^{-10} 2.3×10^{-6}	1.8×10^{-10} 6.8×10^{-7}	2.6×10^{-10} 9.8×10^{-7}	4.1×10^{-10} 7.4×10^{-6}	2.9×10^{-10} 4.5×10^{-7}	5.5×10^{-12} 5.9×10^{-7}
maximum distance error	2.7×10^{-8} 6.9×10^{-5}	8.3×10^{-9} 4.2×10^{-5}	1.2×10^{-5} 2.1×10^{-4}	9.7×10^{-7} 9.1×10^{-4}	1.7×10^{-5} 1.7×10^{-3}	2.0×10^{-8} 1.9×10^{-5}
1000X error reduction as a side effect	best performance improvement			worst performance improvement		

Four (trivial) contributions

1. Raycentric coordinate system
   - Using a sole d.o.f. to allow tight clipping
2. Root localization (the Budan etc thing)
   - Optimizing it for Bézier Curves
   - and allowing efficient conversions
3. Splitting only in the parametric domain $t \in [0,1]$
   - Not splitting control points at all
4. Non-linear root finding
   - Ridders method [1979]

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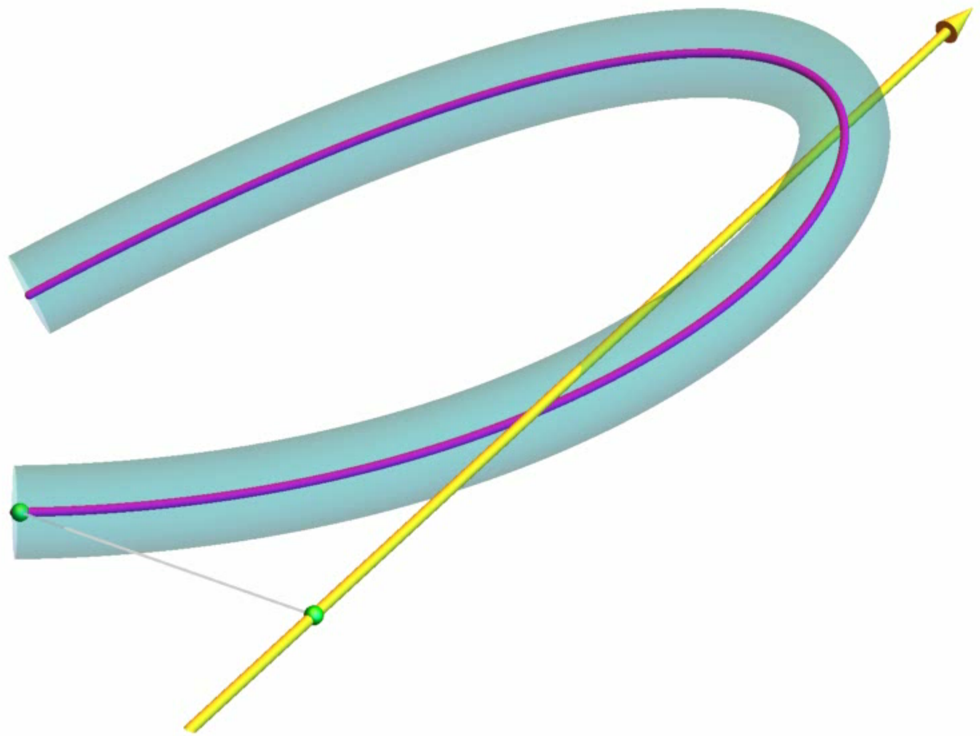
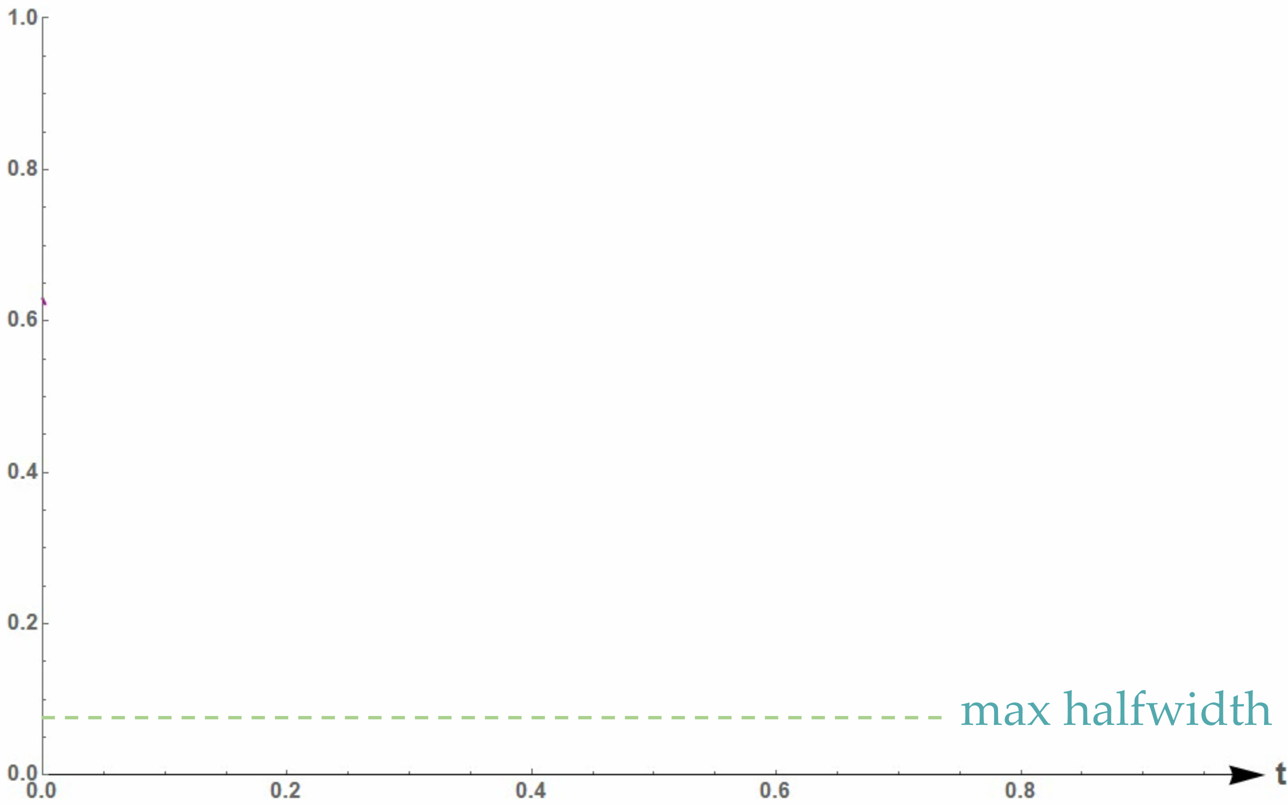
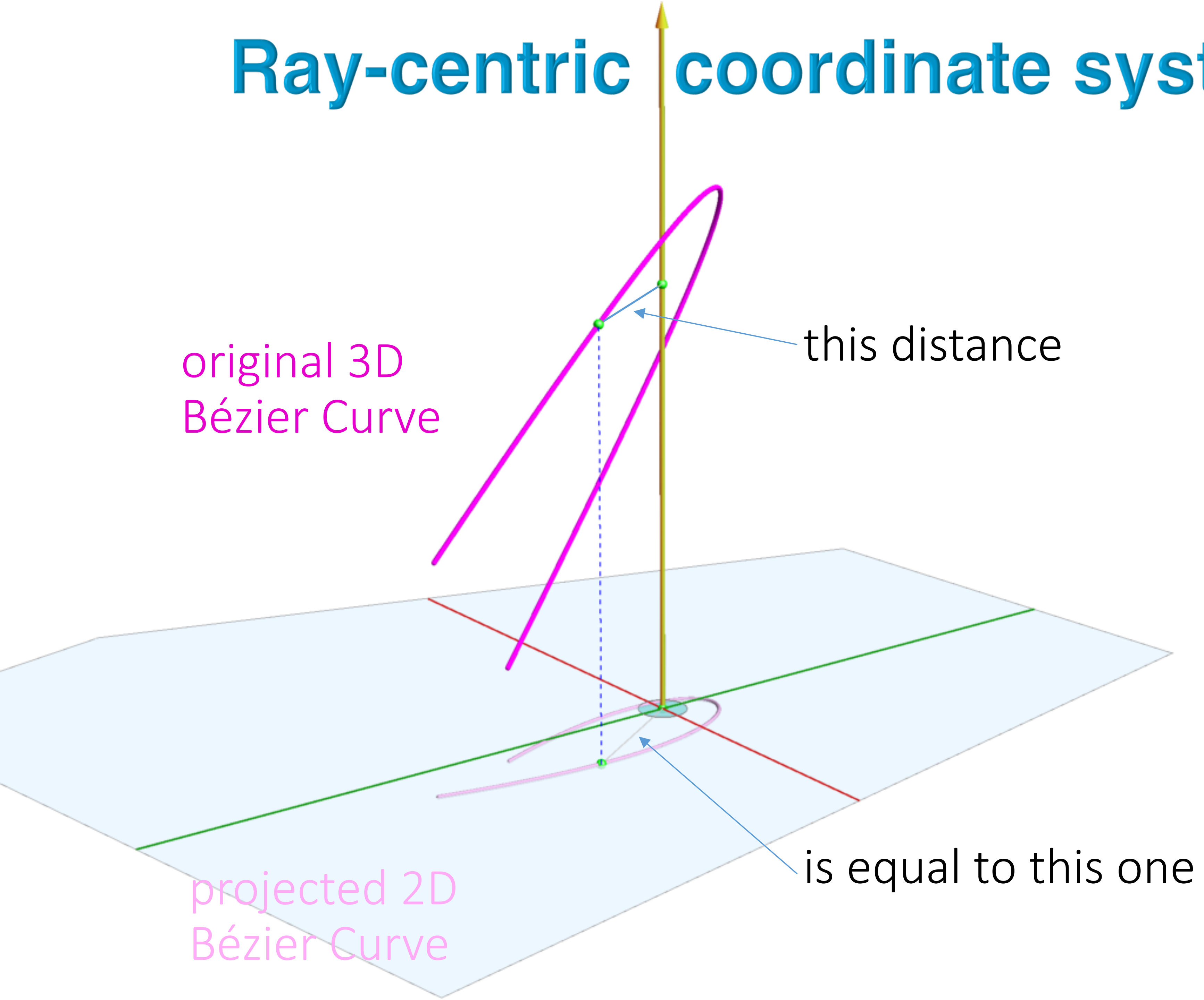
We need minimum of distance(curve(t), ray)

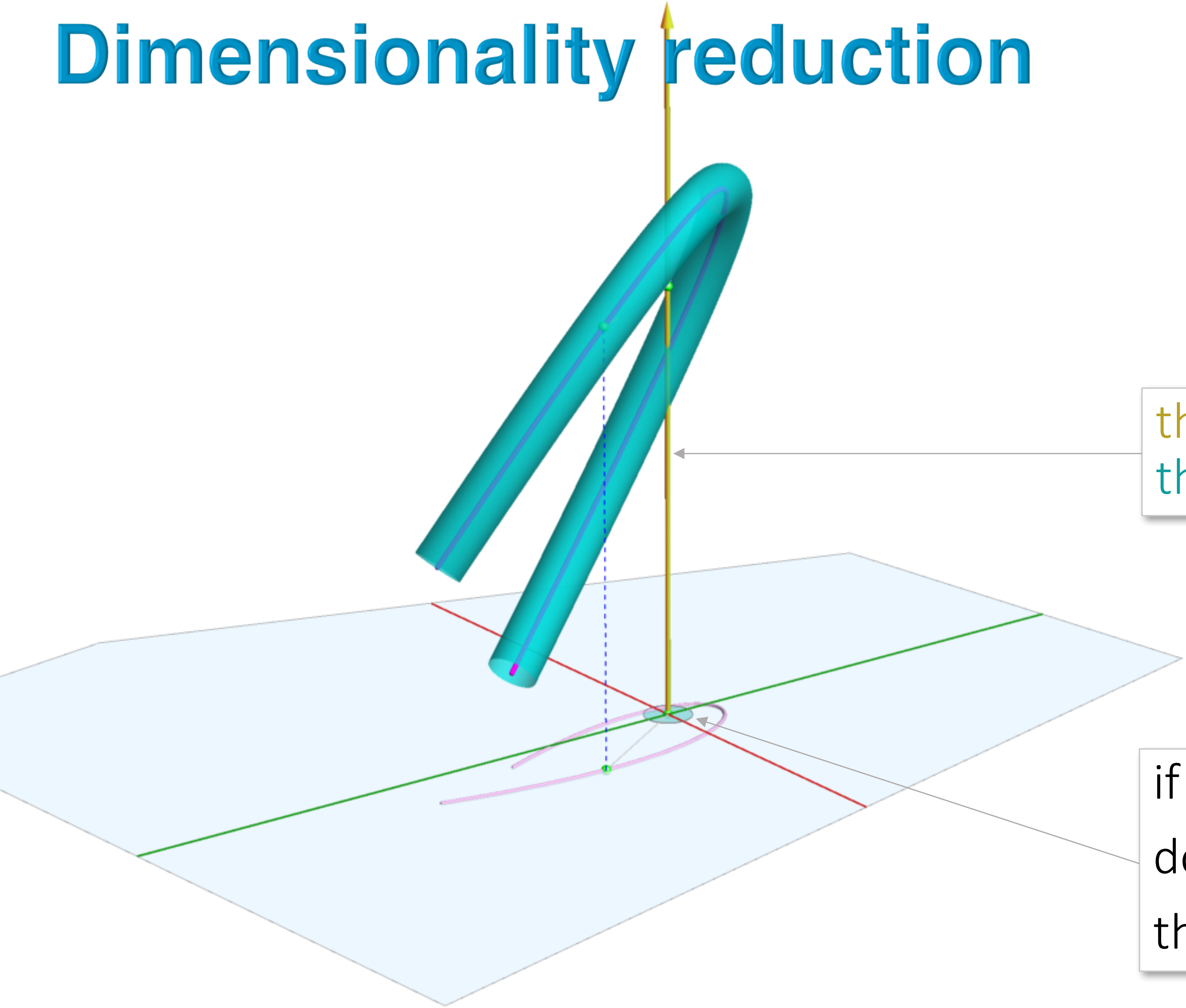




Ray-centric coordinate system



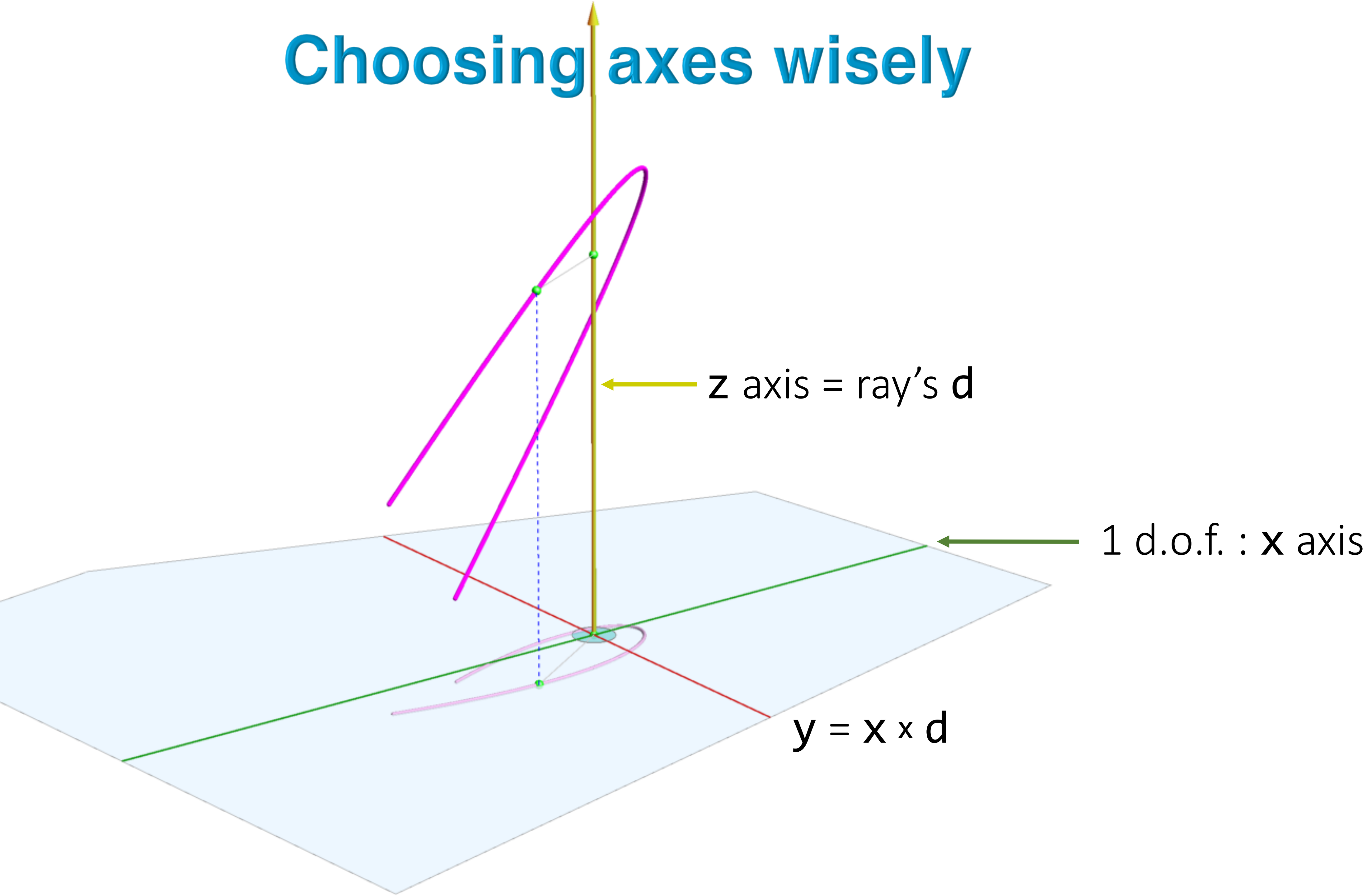
Dimensionality reduction



the ray will not intersect
the swept volume

if the projected curve
does not intersect this circle
then

Choosing axes wisely



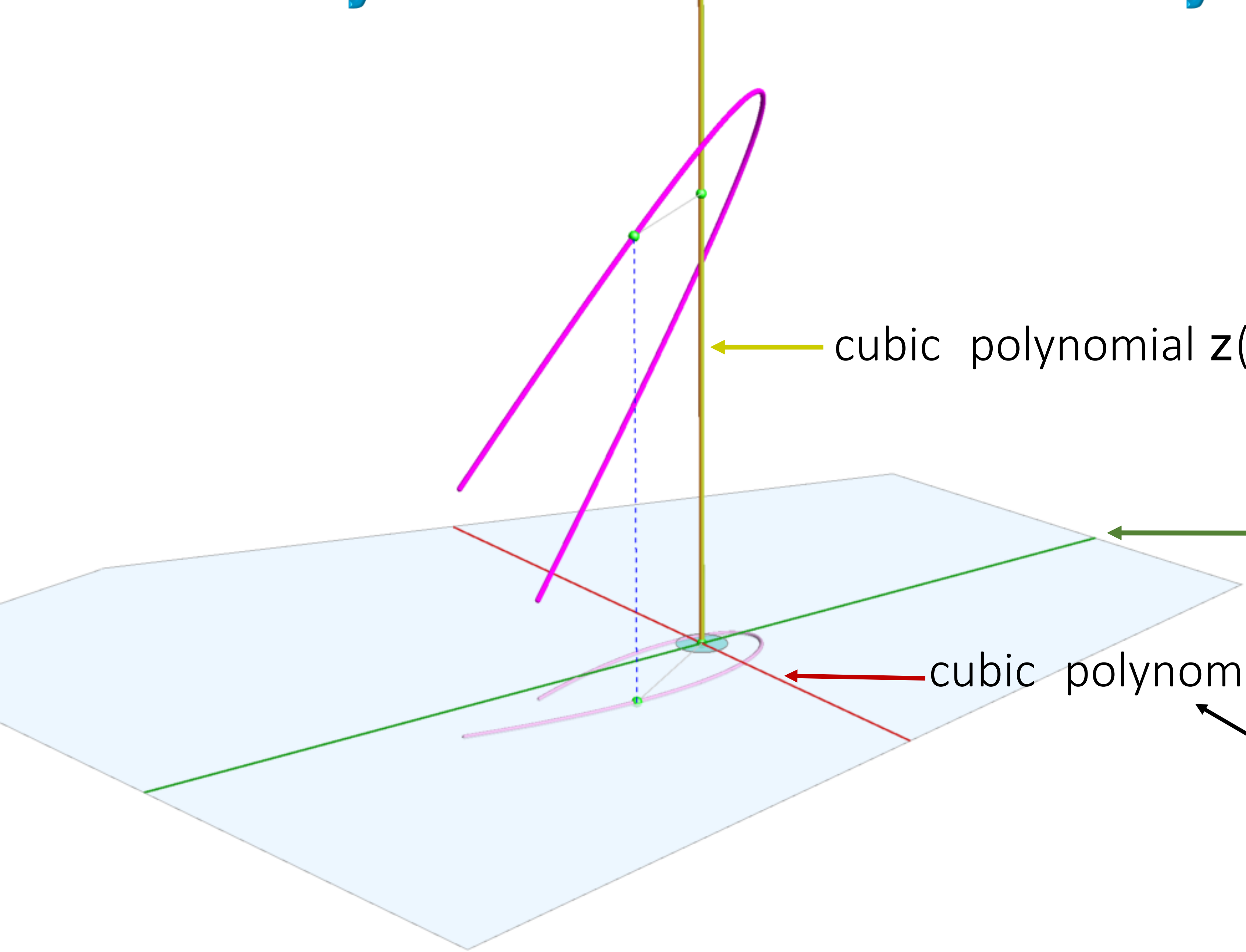
How to choose x axis?


MORE DIFFICULT
LESS DIFFICULT

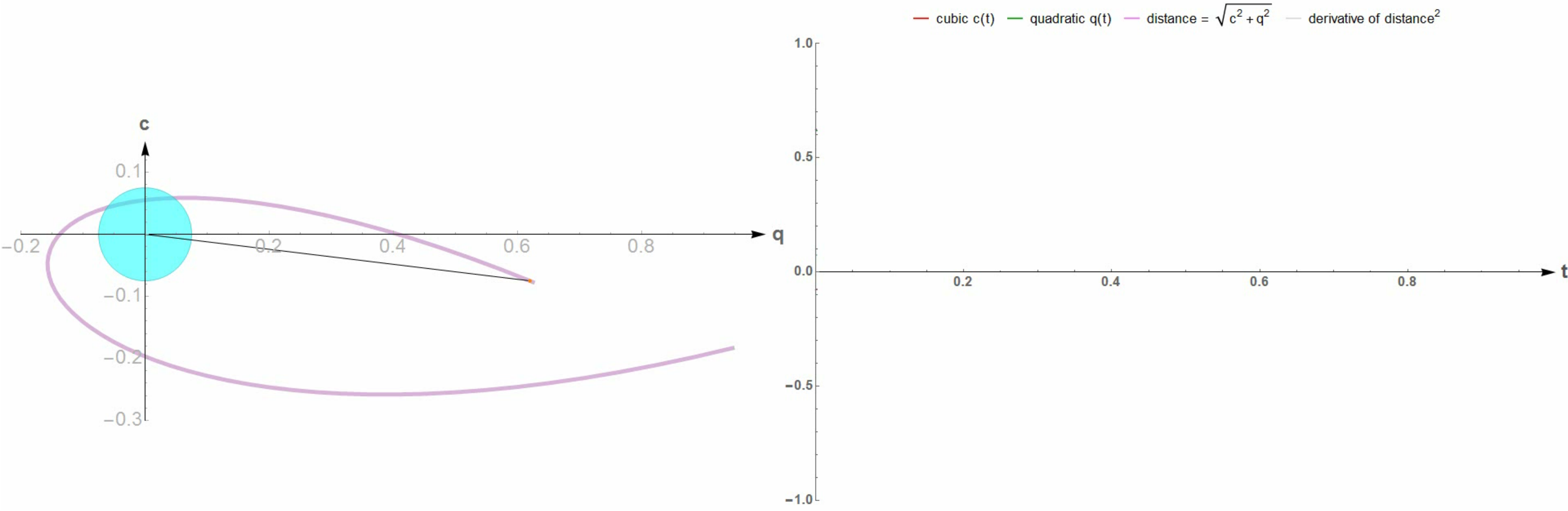
How to choose x axis

it is orthogonal to $\mathbf{d}$	$\mathbf{x}$ = some_vector × $\mathbf{d}$
cubic Bézier curve as a univariate polynomial	$\mathbf{b}(t) = \mathbf{u_0} + \mathbf{u_1}\, t + \mathbf{u_2}\, t^2 + \mathbf{u_3}\, t^3$ where $\mathbf{u_i}$ are 3D vectors and $t \in [0, 1]$
distance from $\mathbf{b}(t)$ to the ray $\mathbf{o} + \mathrm{s}\, \mathbf{d}$ using Πυθαγόρας theorem	$\lvert\mathbf{v}(t)\rvert$, where $\mathbf{v}(t) = (\mathbf{b}(t) - \mathbf{o}) \times \mathbf{d}$ i.e. *cathetus* = *hypotenuse* sin α
x coordinate of $\mathbf{v}(t)$ in $\{\mathbf{x}, \mathbf{y}, \mathbf{d}\}$ orthonormal coordinate system	$\mathbf{v}(t) \cdot \mathbf{x}$
if we choose	$\mathbf{x} = \mathbf{u_3} \times \mathbf{d}$
then the cubic term of $\mathbf{v}(t) \cdot \mathbf{x}$ is	$\mathbf{u_3} \cdot \mathbf{u_3} \times \mathbf{d} = 0$

Ray-centric coordinate system

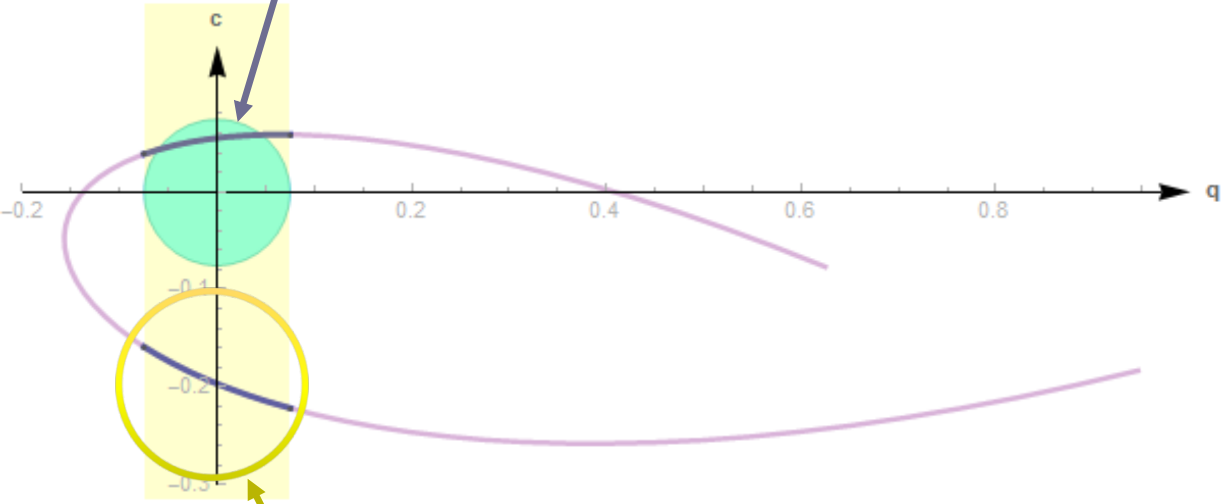


Dual space clipping

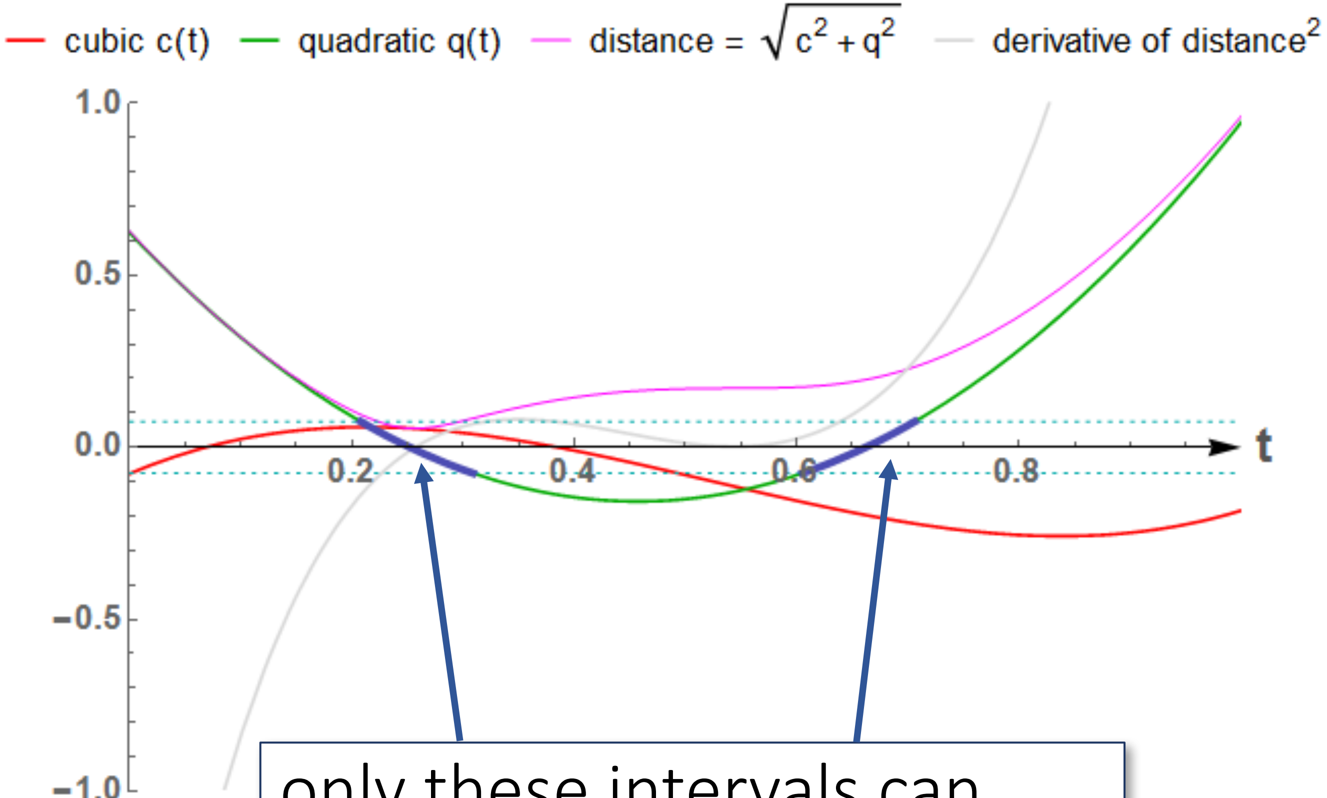


Dual space clipping: interval reduction

and this one can be reduced even further using Budan-Fourier & Vincent's



this interval can be excluded by analyzing the cubic term

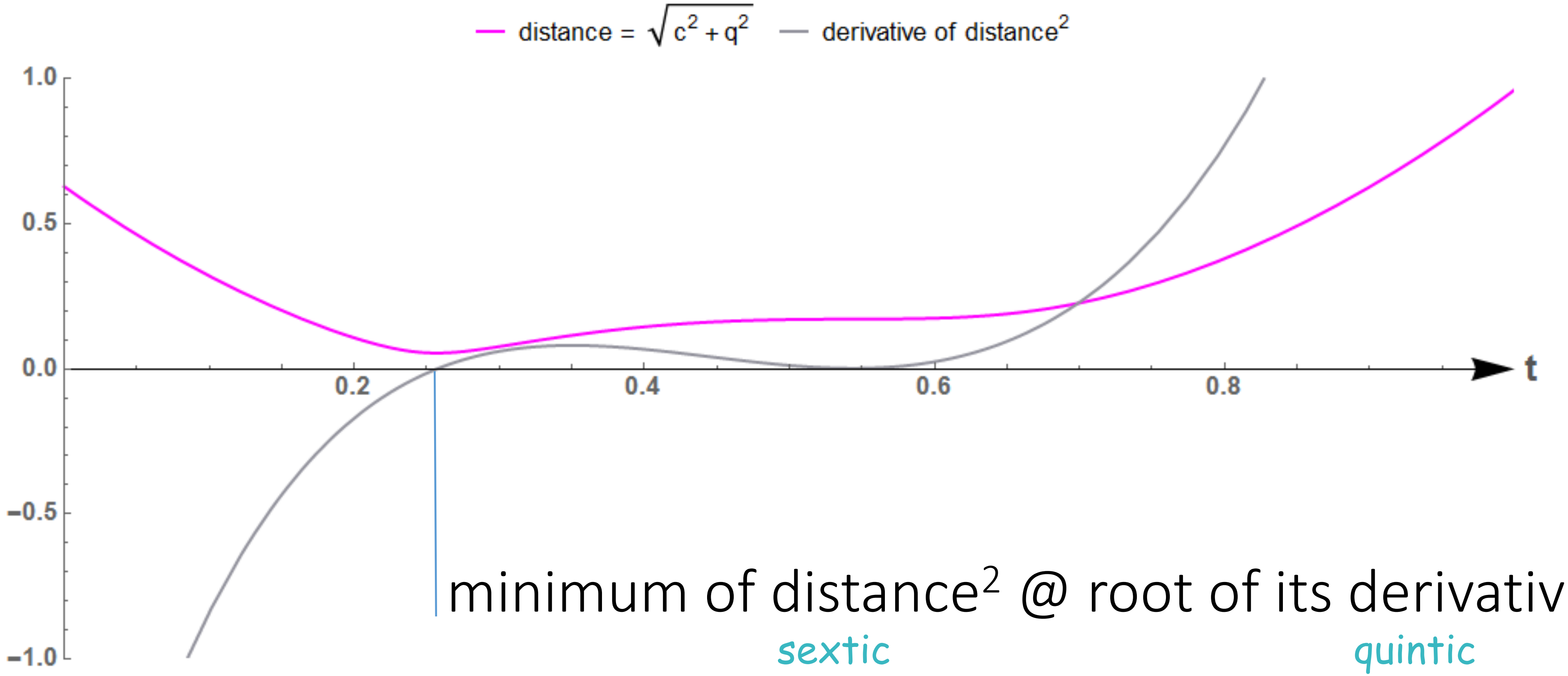


only these intervals can potentially include **min distance < halfwidth**

1. Raycentric coordinate system

- Using a sole d.o.f. to allow tight clipping

2. Root localization (the Budan etc thing)



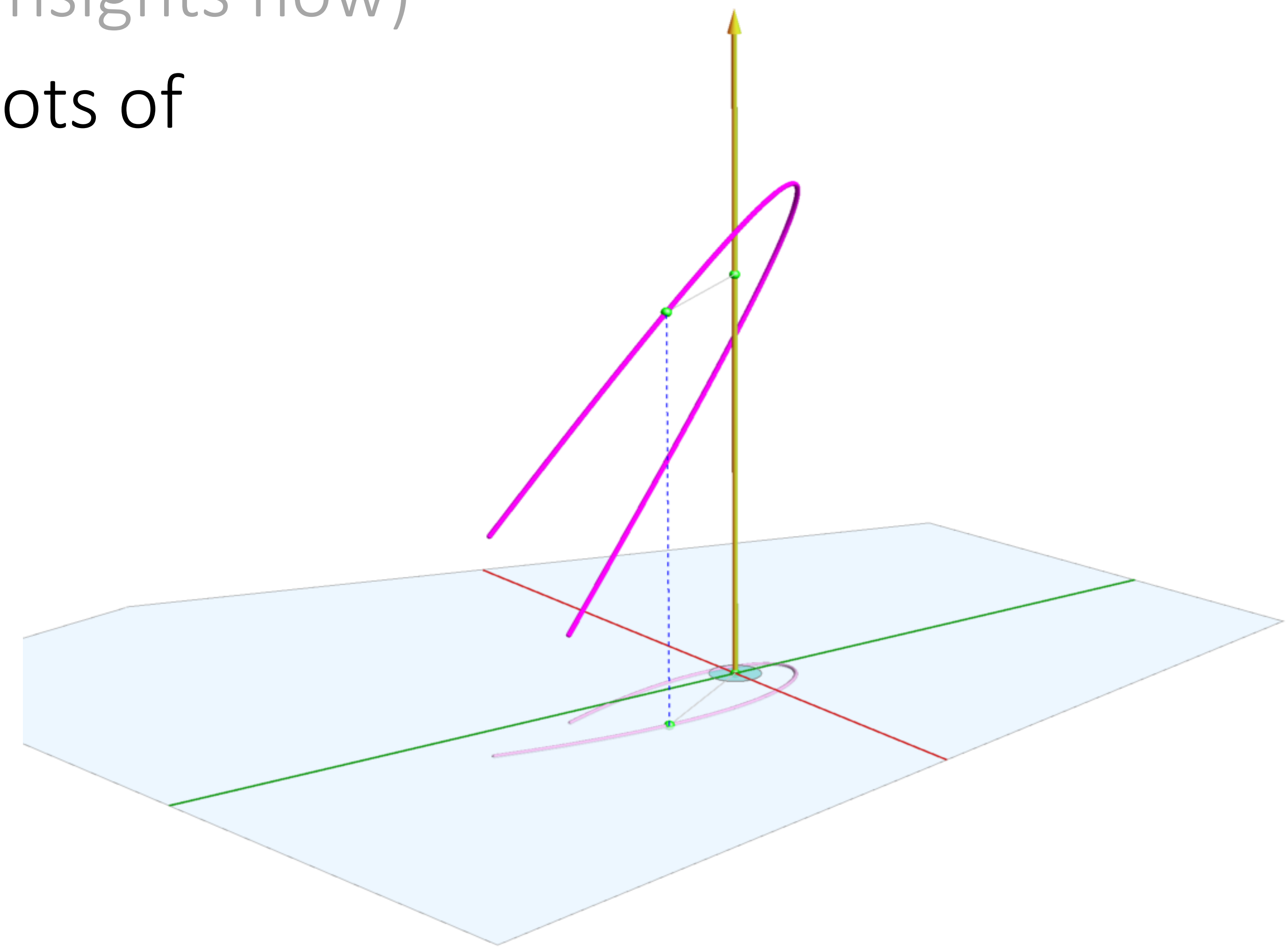
minimum of distance^2 @ root of its derivative

sextic polynomial

quintic polynomial

Details are in the paper…

- plus Mathematica code
  (I just will provide some insights now)
- We want to know # of roots of p(t) for $t \in [t_1, t_2]$



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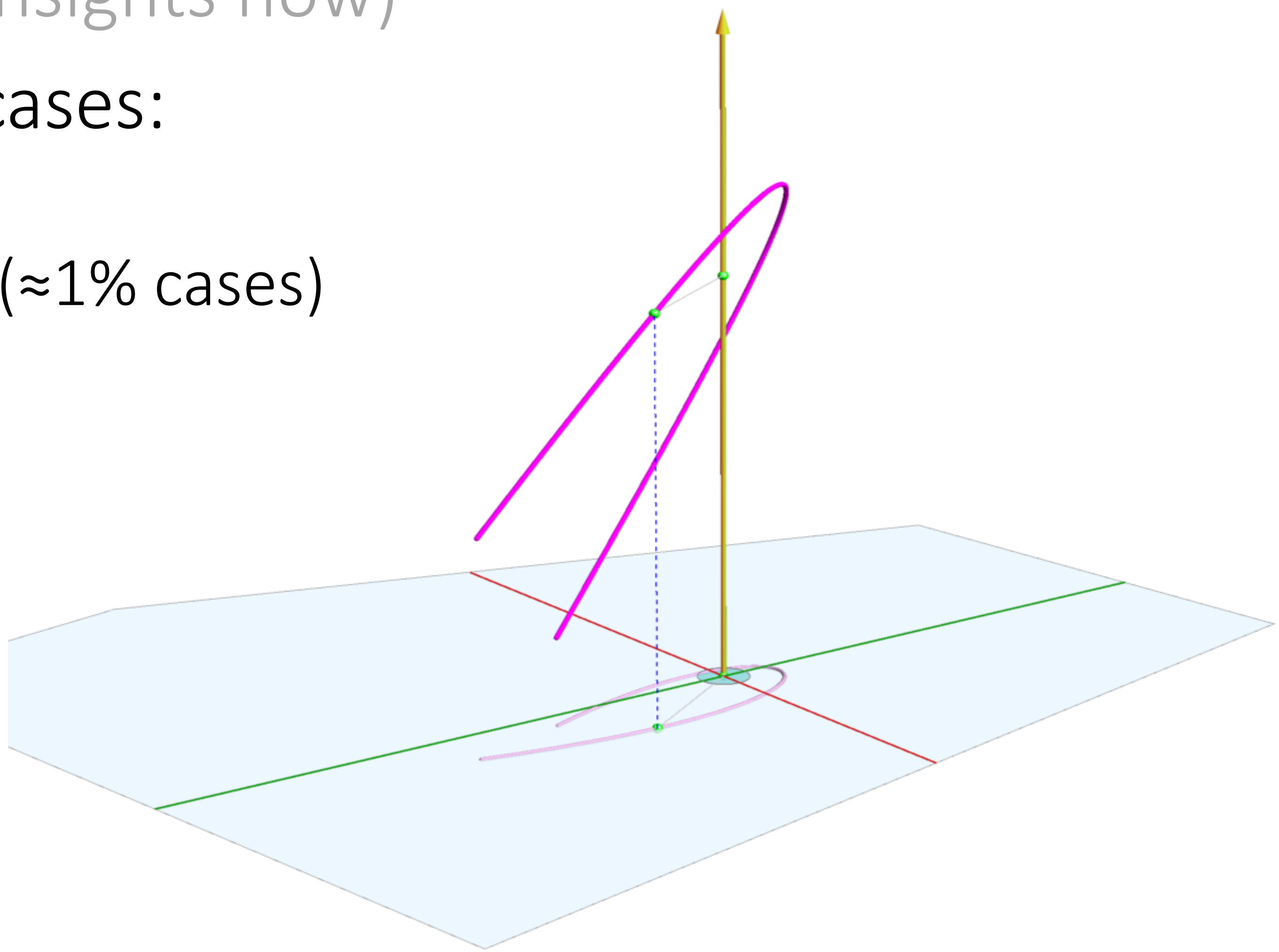
- Only interested in three cases:
  0, 1, or many
  - More than 1 root → split (≈1% cases)



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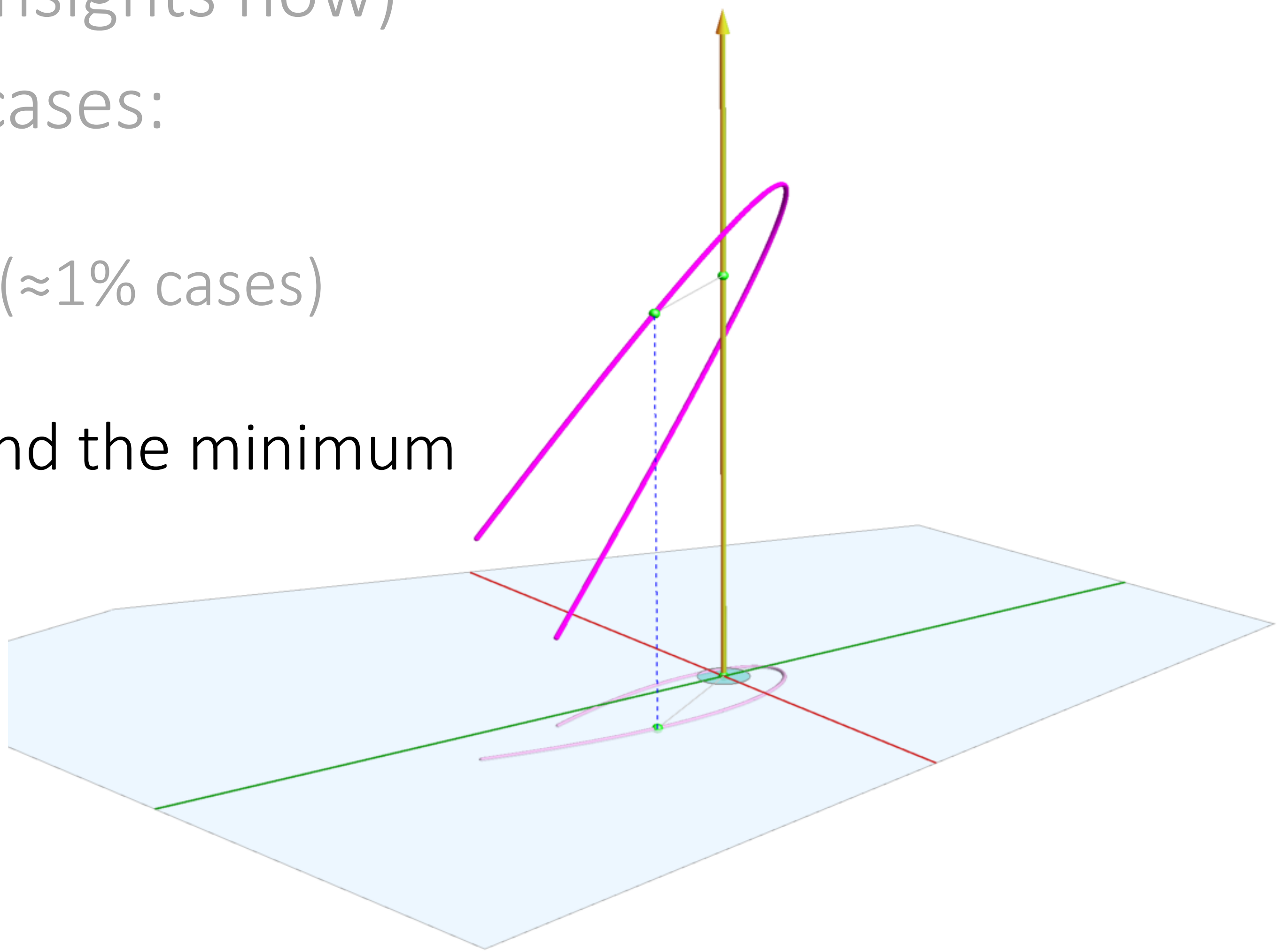
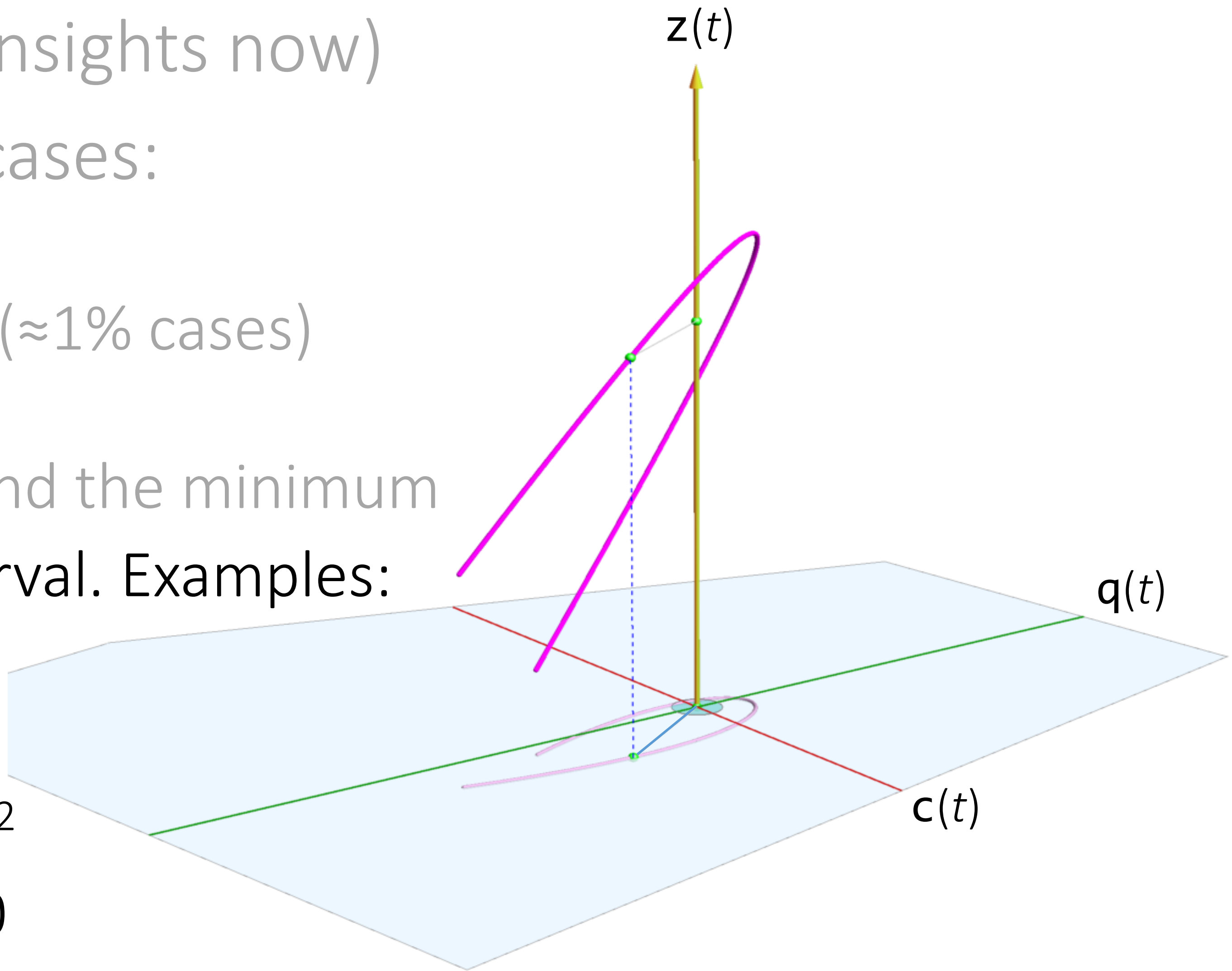
  - One root →
    reduce the interval and find the minimum



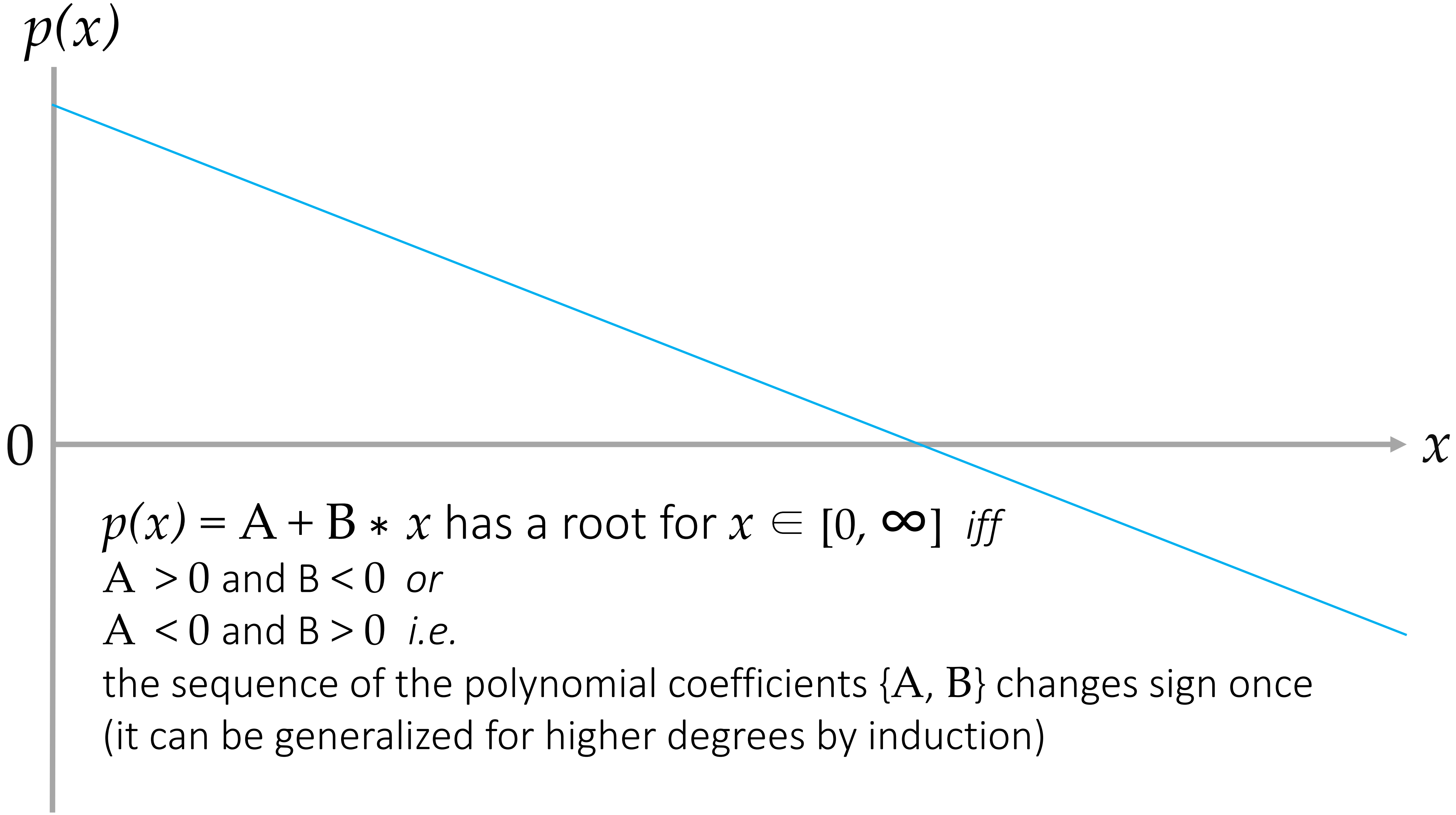
Details are in the paper…

- plus Mathematica code
  (I just will provide some insights now)
- Only interested in three cases:
  0, 1, or many
  - More than 1 root → split (≈1% cases)
  - One root →
    reduce the interval and find the minimum
  - No roots → drop the interval. Examples:

1. $c(t) > |\text{halfwidth}|$
2. $z(t) < 0$
3. $q(t)^2 + c(t)^2 > \text{halfwidth}^2$
4. $q(t)*q'(t) + c(t)*c'(t) \neq 0$



Insights: Descartes' rule of signs



$p(x) = \mathrm{A} + \mathrm{B} * x$ has a root for $x \in [0, \infty]$ *iff*

$\mathrm{A} > 0$ and $\mathrm{B} < 0$ *or*

$\mathrm{A} < 0$ and $\mathrm{B} > 0$ *i.e.*

the sequence of the polynomial coefficients $\{\mathrm{A}, \mathrm{B}\}$ changes sign once

(it can be generalized for higher degrees by induction)

15-16th centuries

- 1637: Descartes' rule of sign
  - # of polynomial roots $\leq$ $\sum$ sign differences between its coefficients
  - no proof, just a few confirming examples
- Wallis' claim: it was proposed by Harriot before
  - 1627–1629: Anglo-French War
- 1707: the rule restated by Newton
  - No proof either
  - Encyclopædia Britannica:
    "he considered its proof too trivial to bother recording"
- 1728 : first (flawed) proof by Segner (written in Latin)
- 1742: first real proof by De Gua

The problem: we need $[a, b]$ and not $[0, \infty]$

- 1807: Budan's theorem

  $p(x)$ has no real roots in [a, a+1]
  if $p(a)$ and $p(a+1)$ have the same number of sign variations

- 1820: Fourier's theorem

  compute # of sign variations lost from *a* to *b* in
  $p(x)$, $p'(x)$, $p''(x)$,… $p^{(n)}(x)$

equivalent formulations

- 1834: Vincent's theorem

  based on continued fractions
  (these 3 formulations yield upper bound on # of roots)

- 1829: Sturm sequence: exact #, though it requires **l o n g** polynomial division

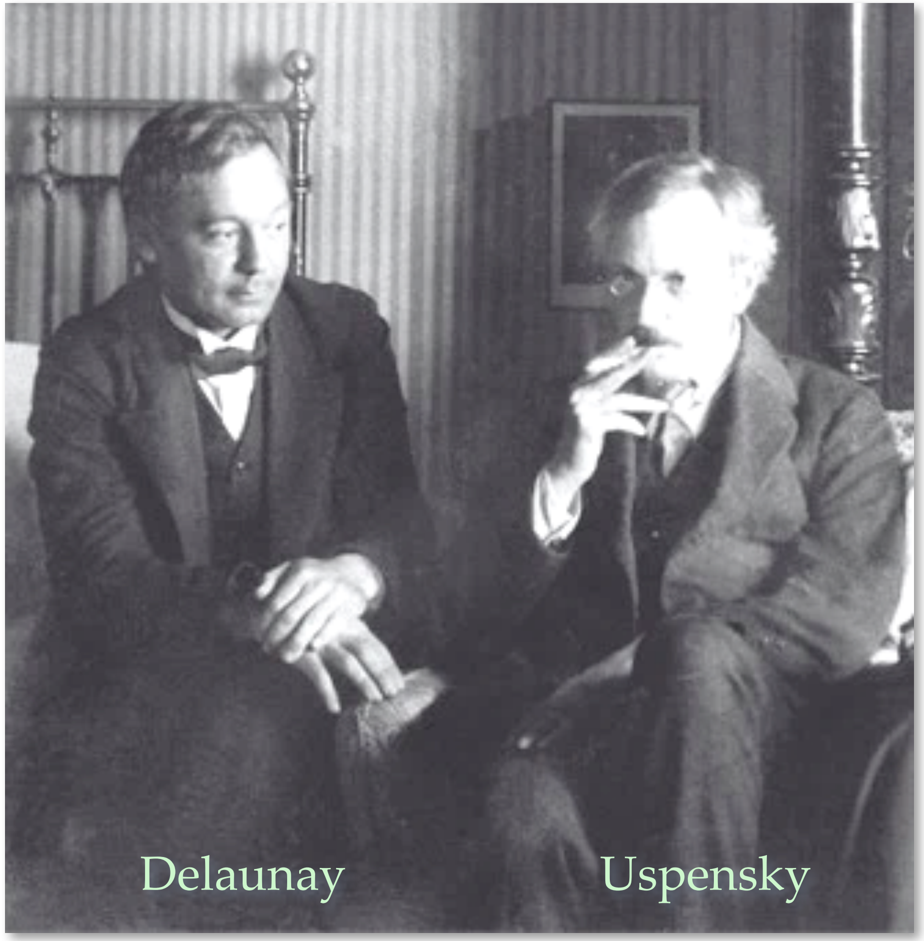
Fast-forward to 20th century

20th century

- 1948: Uspensky's implementation of Vincent's theorem “faster than Sturm’s algorithm”

Historical trivia:

Unfortunately, Uspensky did not have access to Vincent's original paper of 1836, in which Budan’s theorem is cited, and his implementation is actually slower than Sturm’s

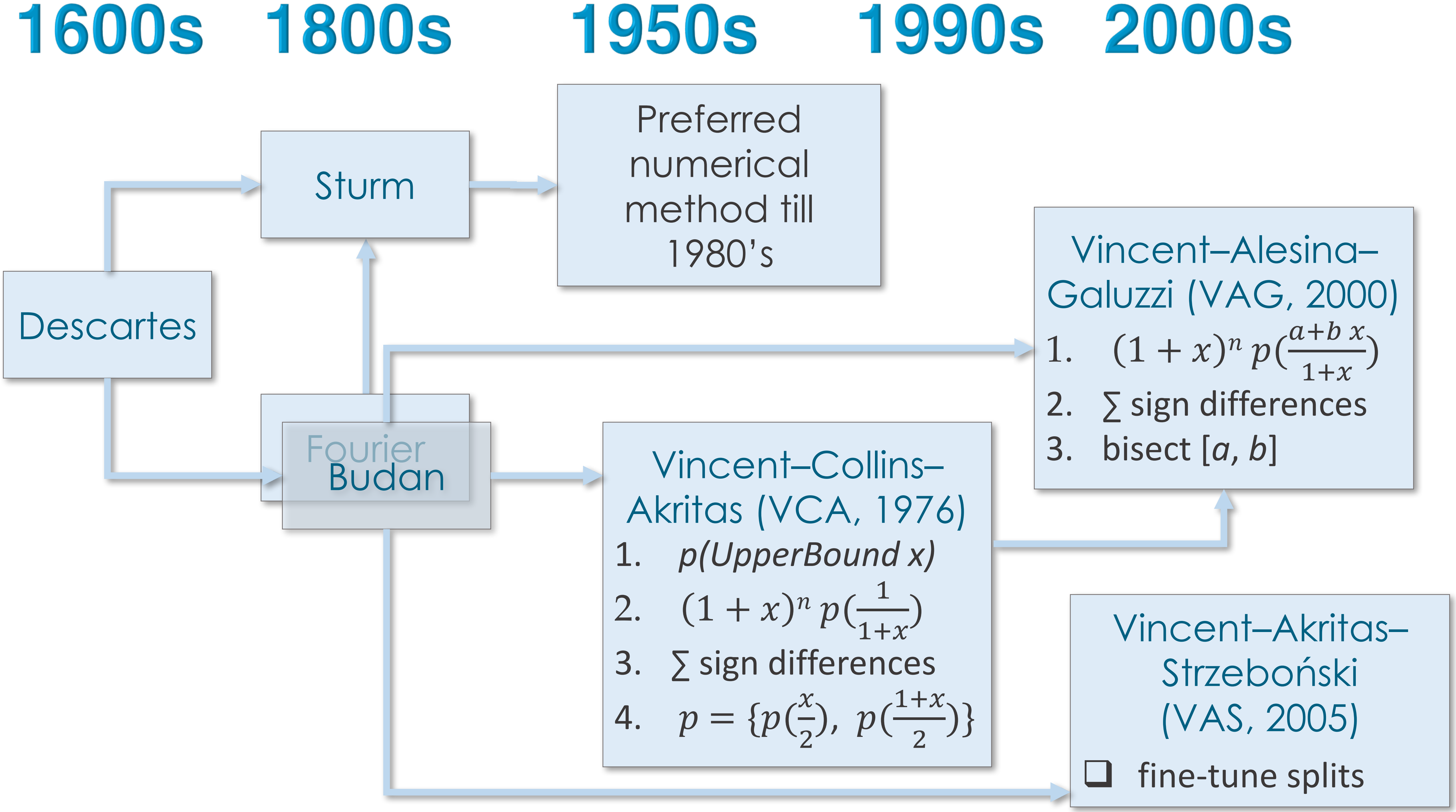


20th century

- 1948: Uspensky's implementation of Vincent's theorem
- 1978: Akritas read about Vincent's theorem in Uspensky's book and made it the topic of his Ph.D. Thesis "Vincent's Theorem in Algebraic Manipulation"

20th century

- 1948: Uspensky's implementation of Vincent's theorem
- 1978: Akritas found Vincent's theorem in Uspensky's book and made it the topic of his Ph.D. Thesis "Vincent's Theorem in Algebraic Manipulation“
- Fortunately for Akritas, the heroic librarian at the University of Wisconsin was able to get ahold of the original Vincent's paper


1600s
1800s
1950s
1990s
2000s
Descartes
Sturm
Fourier
Budan
Preferred numerical method till 1980's
Vincent–Alesina–Galuzzi (VAG, 2000)
1. $(1+x)^n\, p(\frac{a+b\,x}{1+x})$
2. ∑ sign differences
3. bisect [a, b]
Vincent–Collins–Akritas (VCA, 1976)
1. p(UpperBound x)
2. $(1+x)^n\, p(\frac{1}{1+x})$
3. ∑ sign differences
4. $p = \{p(\frac{x}{2}),\ p(\frac{1+x}{2})\}$
Vincent–Akritas–Strzeboński (VAS, 2005)
fine-tune splits

What is important for us

- Vincent–Collins–Akritas form $(1+x)^n\, p(\frac{1}{1+x})$ for $x \in [0,1]$ is easy to compute for Bézier curves in Bernstein form
- yet we already have a reduced interval [*a*,*b*] and Vincent–Alesina–Galuzzi $(1+x)^n\, p(\frac{a+b\,x}{1+x})$ is somewhat more challenging

What we can do though (see the paper)

- Compute Fourier sequences a_i and b_i for $x = 0$ and $x = 1$
- And then on interval [*a*,*b*] using the chain differentiation rule
- Convert to VAG representation for the derivative of distance^2 as (VAG sequence for distance^2 is computed as partial sums)

$$\begin{pmatrix} b_1 \\ 5b_1 - b_2 \\ 10b_1 - 4b_2 + \frac{b_3}{2} \\ 10a_1 + 4a_2 + \frac{a_3}{2} \\ 5a_1 + a_2 \\ a_1 \end{pmatrix}$$

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3. **Splitting only in the parametric domain $t \in [0,1]$**
   - Not splitting control points at all
4. Non-linear root finding
   - Ridders method [1979]

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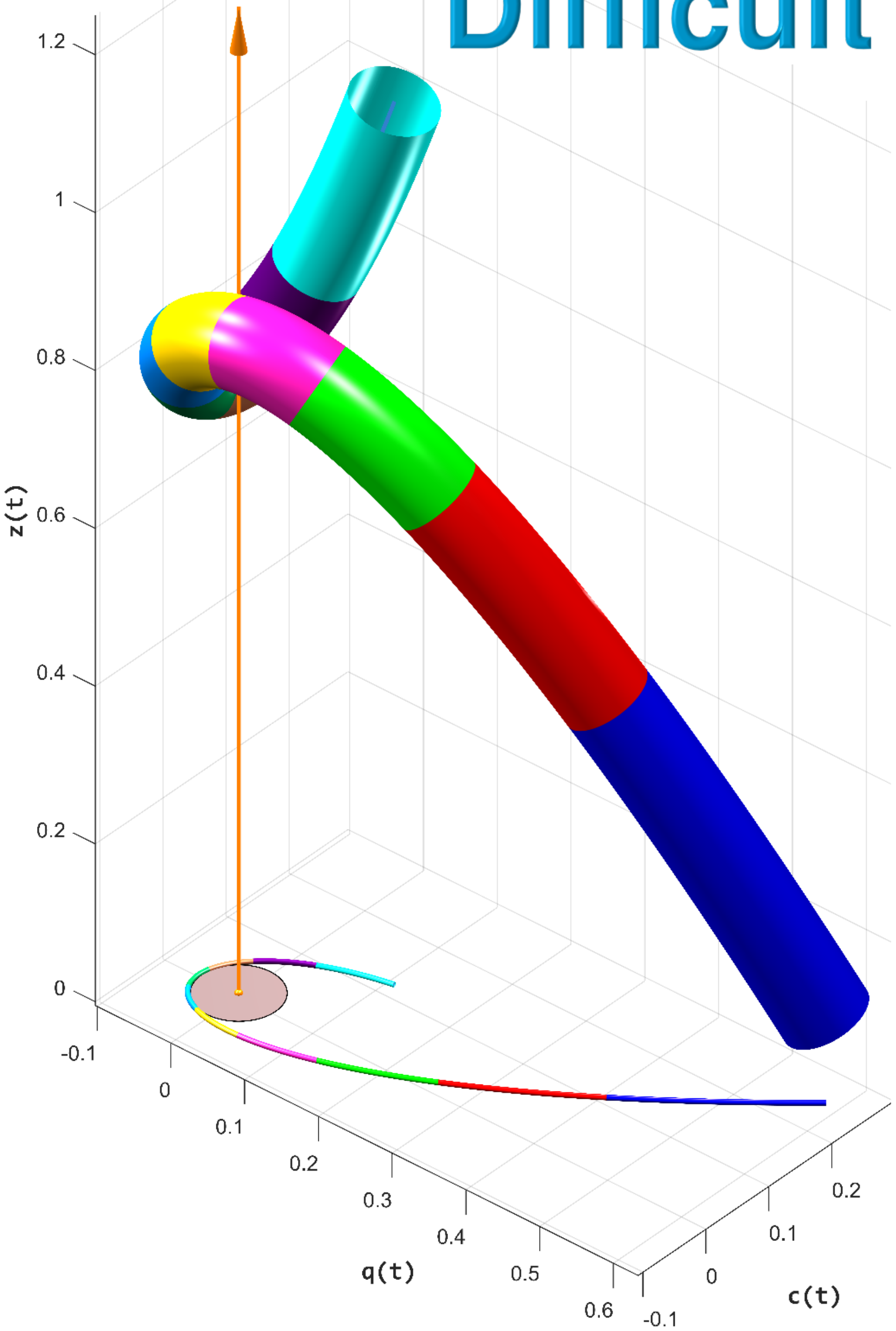
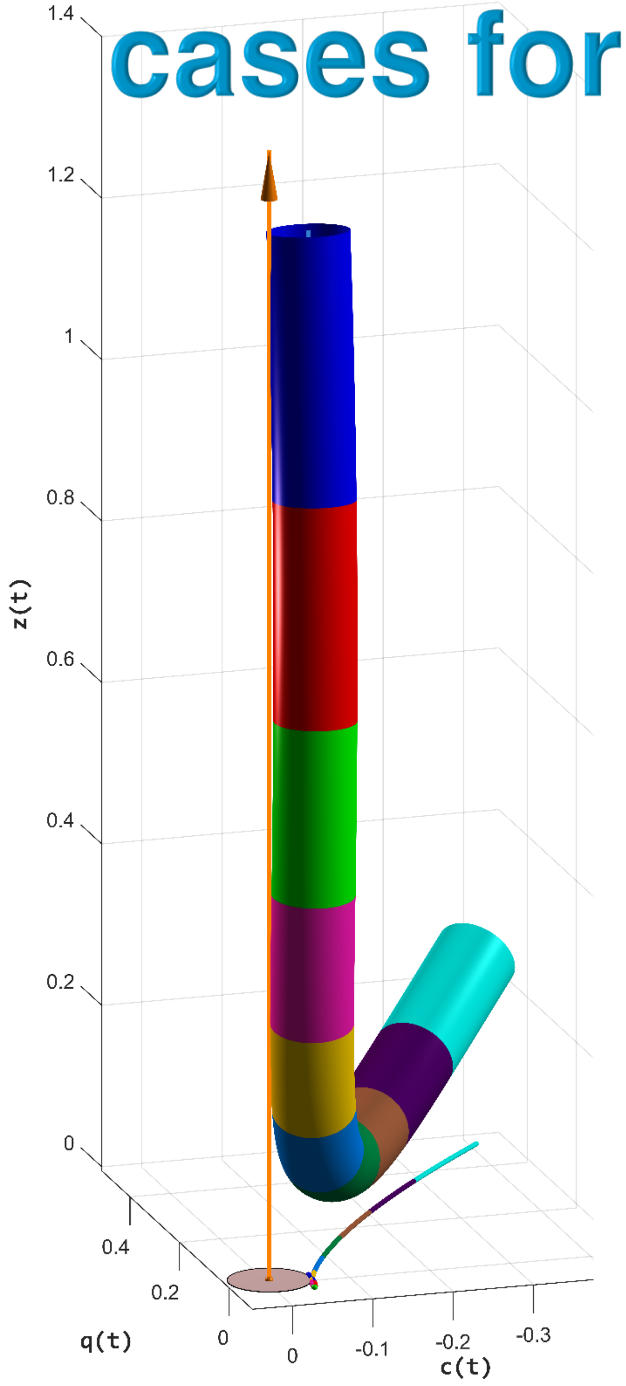
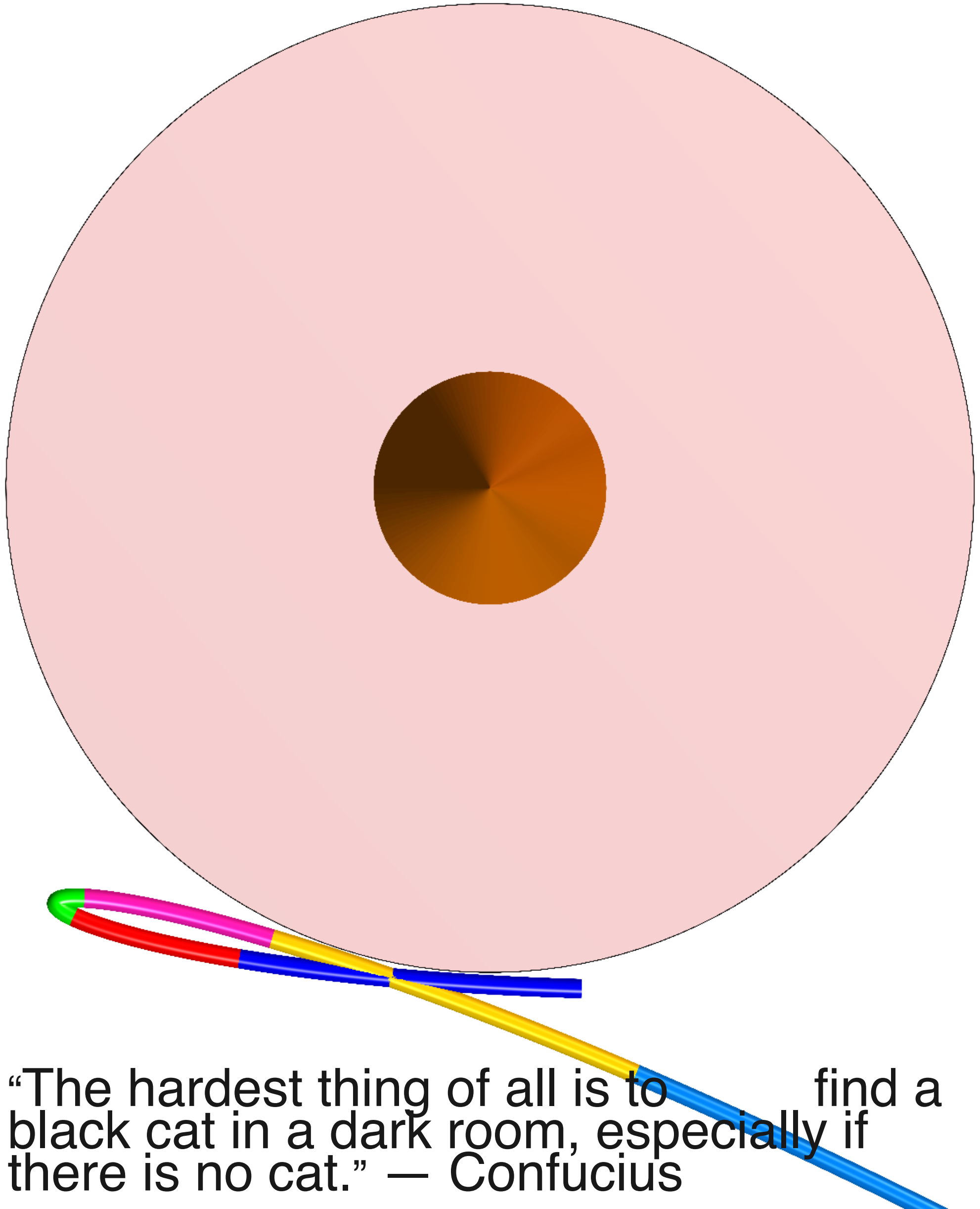
4. Non-linear root finding

Go over 37 entries at https://en.wikipedia.org/wiki/Category:Root-finding_algorithms and choose the best one (see the paper for the discussion)

Difficult cases for linear methods

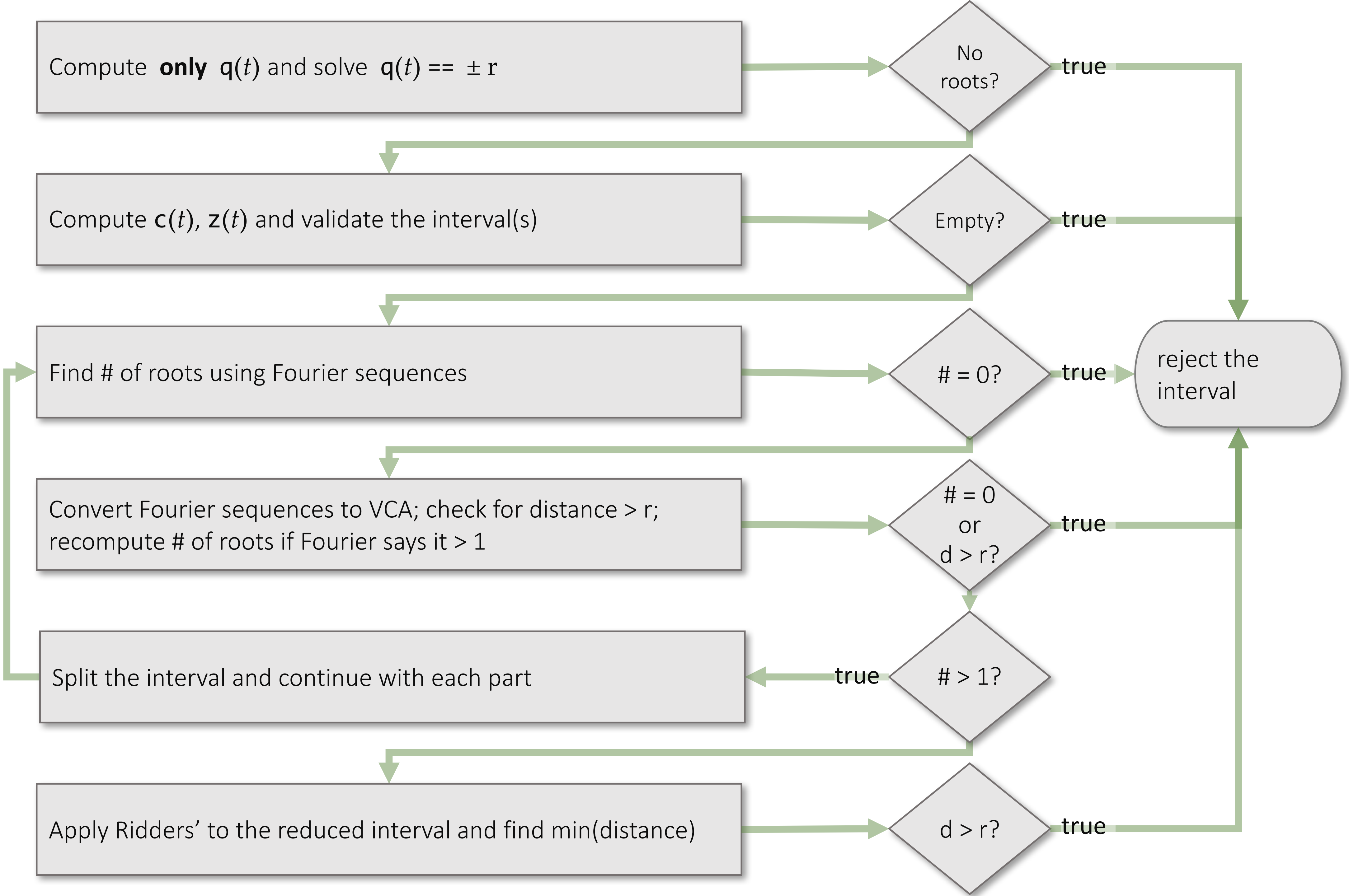






"The hardest thing of all is to find a black cat in a dark room, especially if there is no cat." — Confucius

Summary


Compute **only** $q(t)$ and solve $q(t) == \pm r$
No roots?
true
Compute $c(t)$, $z(t)$ and validate the interval(s)
Empty?
true
Find # of roots using Fourier sequences
= 0?
true
reject the interval
Convert Fourier sequences to VCA; check for distance > r; recompute # of roots if Fourier says it > 1
= 0 or d > r?
true
Split the interval and continue with each part
true
> 1?
Apply Ridders' to the reduced interval and find min(distance)
d > r?
true

Scripts & backup

How to choose x axis

it is orthogonal to $\mathbf{d}$	$\mathbf{x}$ = some_vector × $\mathbf{d}$
cubic Bézier curve as a univariate polynomial	$\mathbf{b}(\mathrm{t}) = \mathbf{u}_0 + \mathbf{u}_1 t + \mathbf{u}_2 t^2 + \mathbf{u}_3 t^3$ where $\mathbf{u}_\mathbf{i}$ are 3D vectors and $t \in [0, 1]$
distance from $\mathbf{b}(\mathrm{t})$ to the ray $\mathbf{o} + \mathrm{s}\,\mathbf{d}$ using Πυθαγόρας theorem	$\lvert\mathbf{v}(t)\rvert$, where $\mathbf{v}(t) = (\mathbf{b}(t) - \mathbf{o}) \times \mathbf{d}$ i.e. cathetus = hypotenuse sin α
x coordinate of $\mathbf{v}(t)$ in $\{\mathbf{x}, \mathbf{y}, \mathbf{d}\}$ coordinate system	$\mathbf{v}(t) \cdot \mathbf{x}$
if we choose	$\mathbf{x} = \mathbf{u}_3 \times \mathbf{d}$
then cubic term of $\mathbf{v}(t) \cdot \mathbf{x}$ is	$\mathbf{u}_3 \cdot \mathbf{u}_3 \times \mathbf{d} = 0$

(*at least*) two approaches

% std::istringstream("1685204011 2815682195 385126455 355161270 2741490130 2213418773 3437980441 3135777609 3486594975 1234267285 1099369070 1004897494 1602088570
2058086804 2766595287 3177268504 2702257490 3124146337 1418718049 1365130483 1265208718 1884250240 3648855035 639353071 4256957795 1605271289 982700018
1959955605 207137408 4223255709 3268747380 62291474 720723770 1432001539 1179295468 4277700772 3384822023 789312185 2745877539 1044122156 2253754244 1044010503
322875373 4015799724 1345322295 99273851 2020442037 2323035699 3484228229 3559337231 4125372910 2067433396 44722077 1341711910 2137959577 1496281135 201449566
2288112463 1808241307 1298612477 3798433704 1473569204 465168807 1173655074 545378063 1215215660 3648634627 759126851 2225790131 2086325018 2554369261 2276960658
941880856 1066069953 1717952623 1984850353 3343835836 2695933096 1656822112 3932875924 3773638371 931018417 2598849831 2643279084 3503712314 132947922 3097522224
3702851089 3056142194 1195040649 412219650 1994870821 337765008 93156141 95192607 1336967578 3577484679 2396266869 4258980693 3395204945 582432190 1581048411
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Why *t*?

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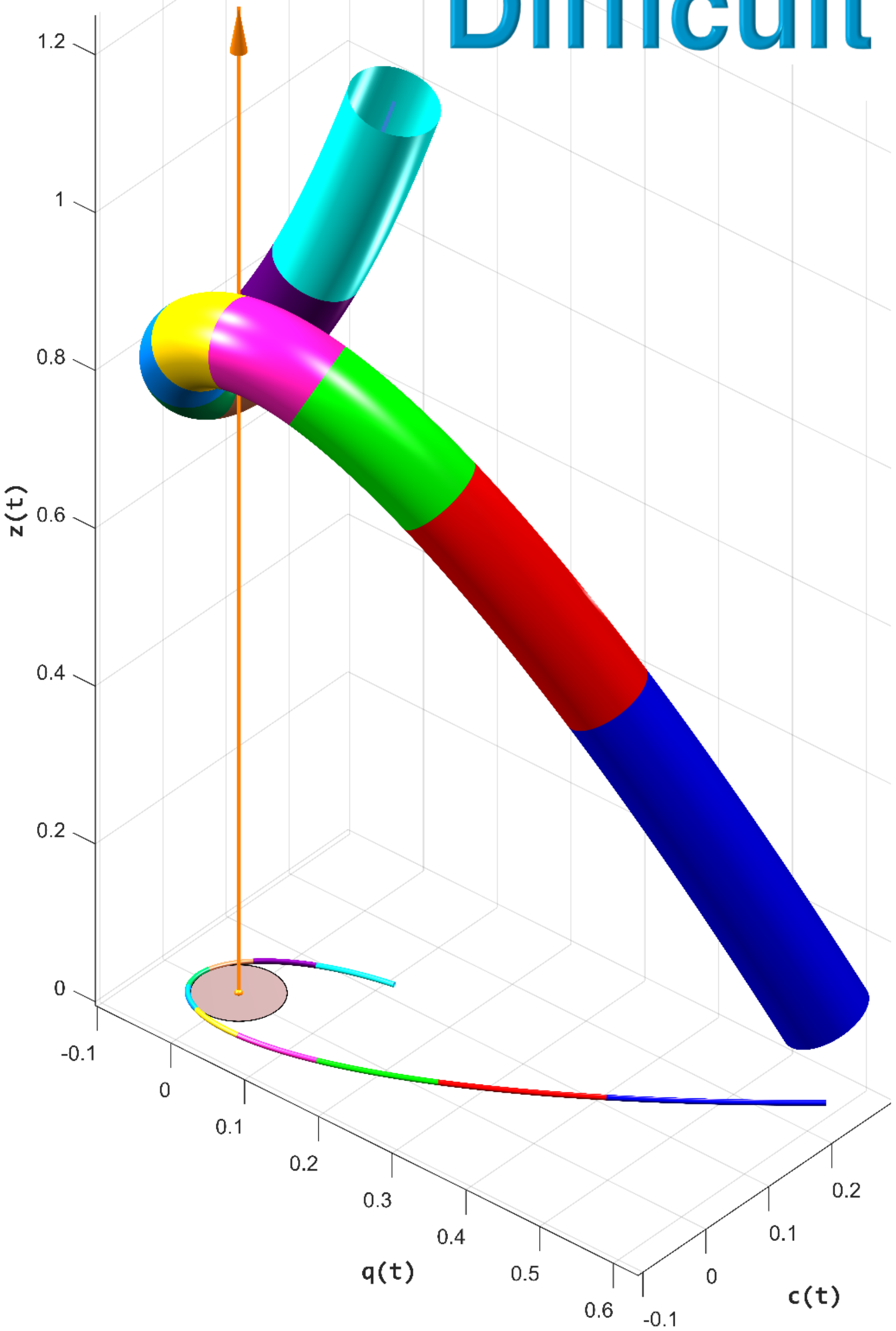
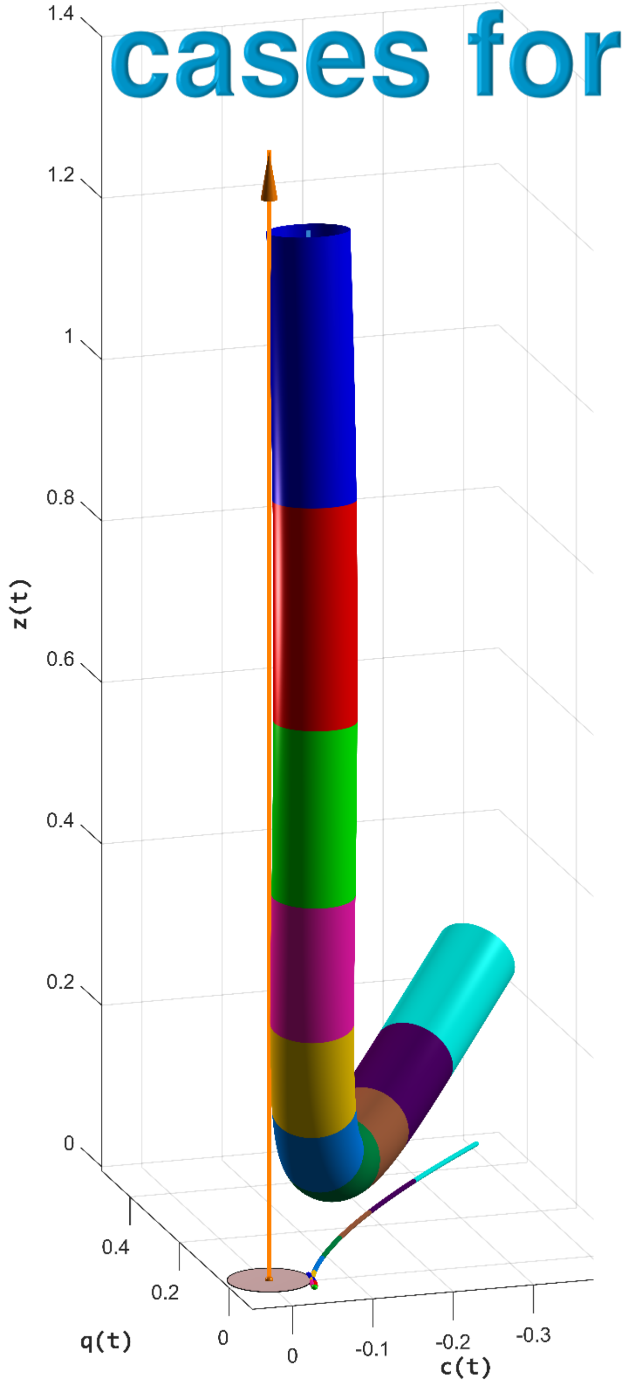
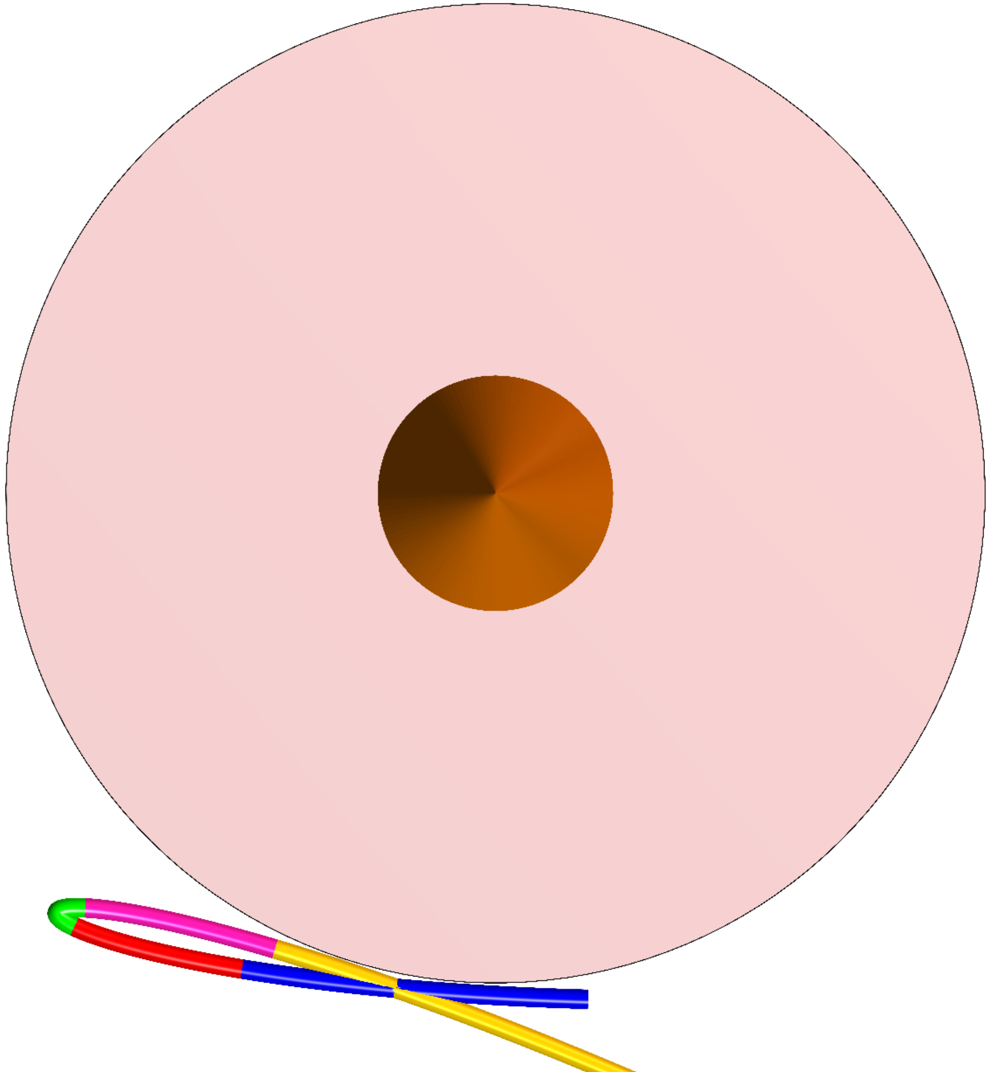
Difficult cases for linear methods







“The hardest thing of all is to find a black cat in a dark room, especially if there is no cat.” — Confucius