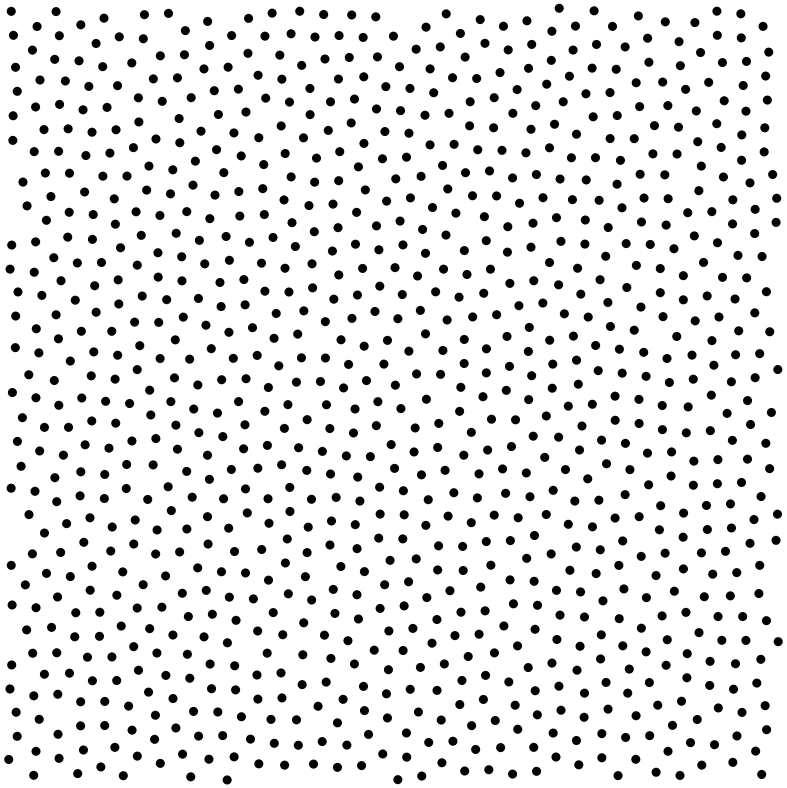
Farthest-Point Optimized Point Sets with Maximized Minimum Distance

Thomas Schlömer Daniel Heck Oliver Deussen

Computer Graphics Group
University of Konstanz

Introduction

Well-Distributed Point Sets



Uniform

- Avg. point density approx. constant
- No “holes” or clusters

Introduction

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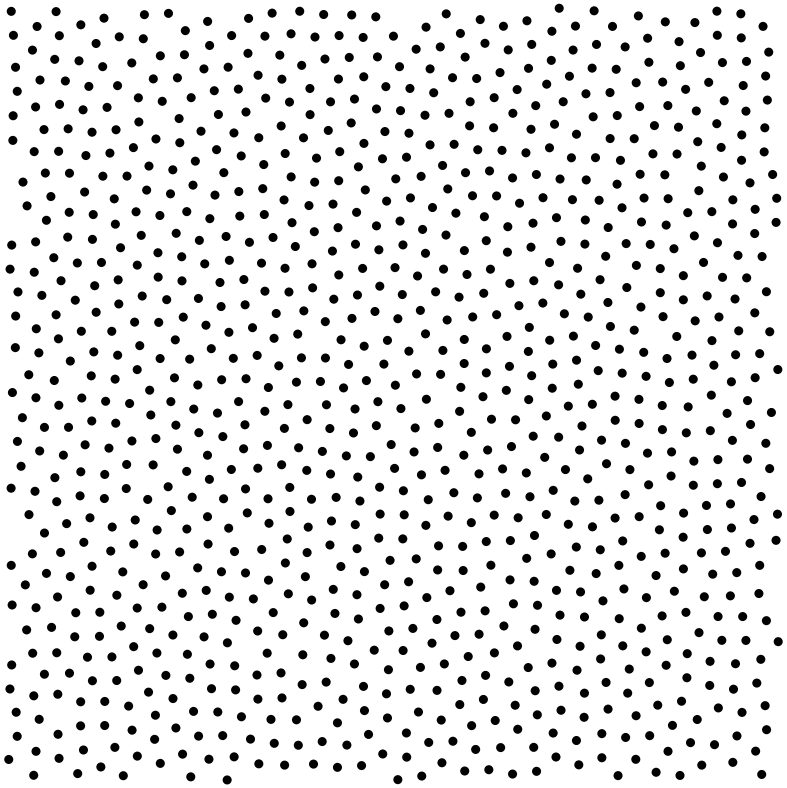

Uniform

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- No “holes” or clusters

1. High minimum distance
   Prevents “clumping”
2. Irregularity
   Prevents correlations

(1) and (2) combined: “blue noise”

Introduction

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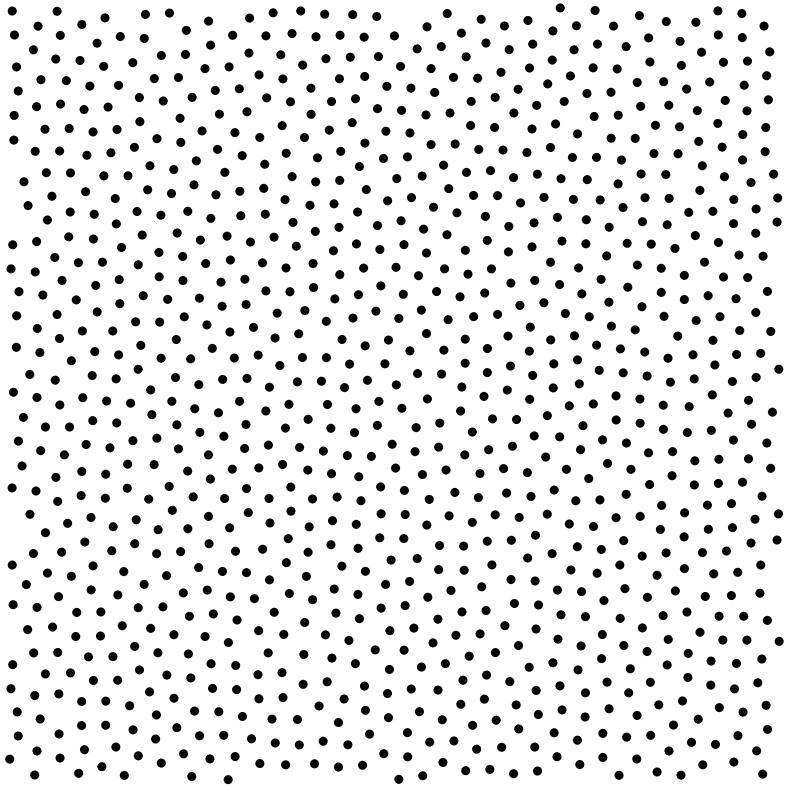
1. Farthest-Point Optimization

Main Algorithm



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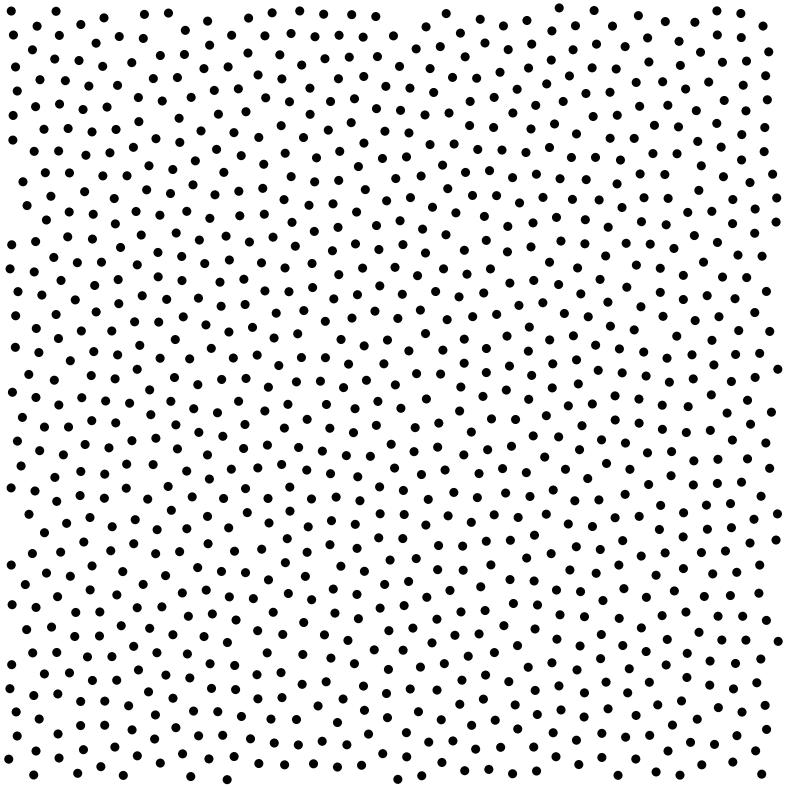
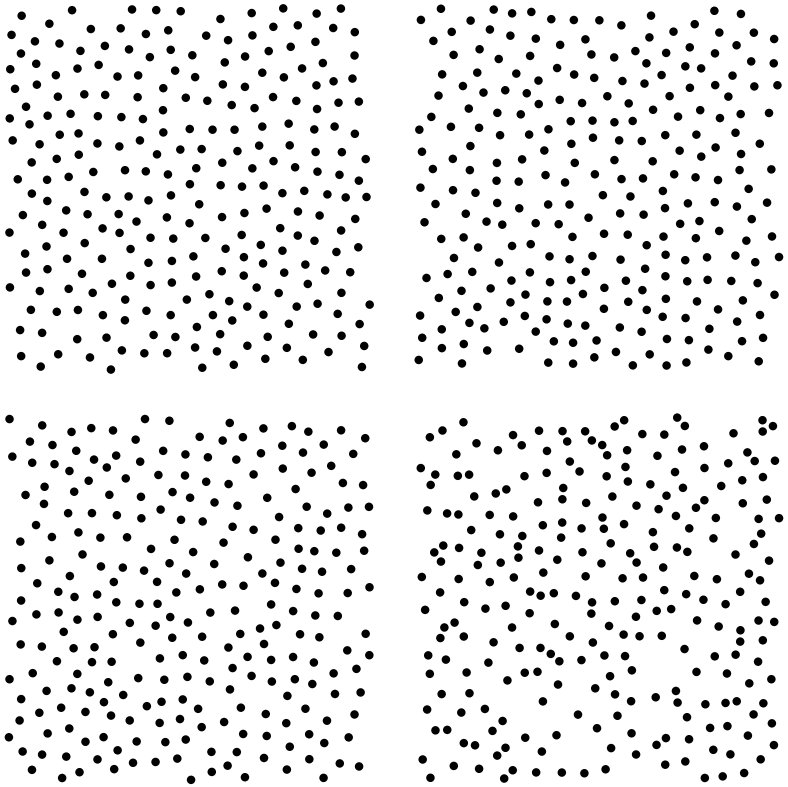


2. Extension

Partition Algorithm



Background

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Measures

- Interested in points in 2D unit square with toroidal metric $d_T(x, y)$
- For a set of points X with $n := |X|$ points we define

$$d_x := \min_{y \in X \setminus \{x\}} d_T(x, y) \qquad \text{local mindist,}$$

$$d_X := \min_{x, y \in X, x \neq y} d_T(x, y) \qquad \text{global mindist,}$$

$$\bar{d}_X := \frac{1}{n} \sum_{x \in X} d_x \qquad \text{average mindist.}$$

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- Largest mindist by hexagonal arrangement $d_{\text{max}} = \sqrt{2/\sqrt{3}n}$
- Normalize every measure by this value

$$\delta_x := d_x / d_{\text{max}}, \quad \delta_X := d_X / d_{\text{max}}, \quad \bar{\delta}_X := \bar{d}_X / d_{\text{max}}.$$

Background

Generation Method	δ_X	$\bar{\delta}_X$	*Note*
Jittered Grid[a]	0.049	0.586	
Best Candidate[b] and FPS[c]	0.751	0.839	
Dart throwing[a] and variants[d]	0.765	0.808	
CCCVT Centroids[e]	0.778	0.896	I
CVT Centroids[f] and			
methods using Lloyd's algorithm	0.795	0.939	I, R
Electrostatic Halftoning[g]	0.826	0.952	I, R
Boundary Sampling[h]	0.829	0.862	
Low discrepancy[i]	0.903	0.920	D, R
Farthest-Point Optimization	*0.930*	*0.932*	I

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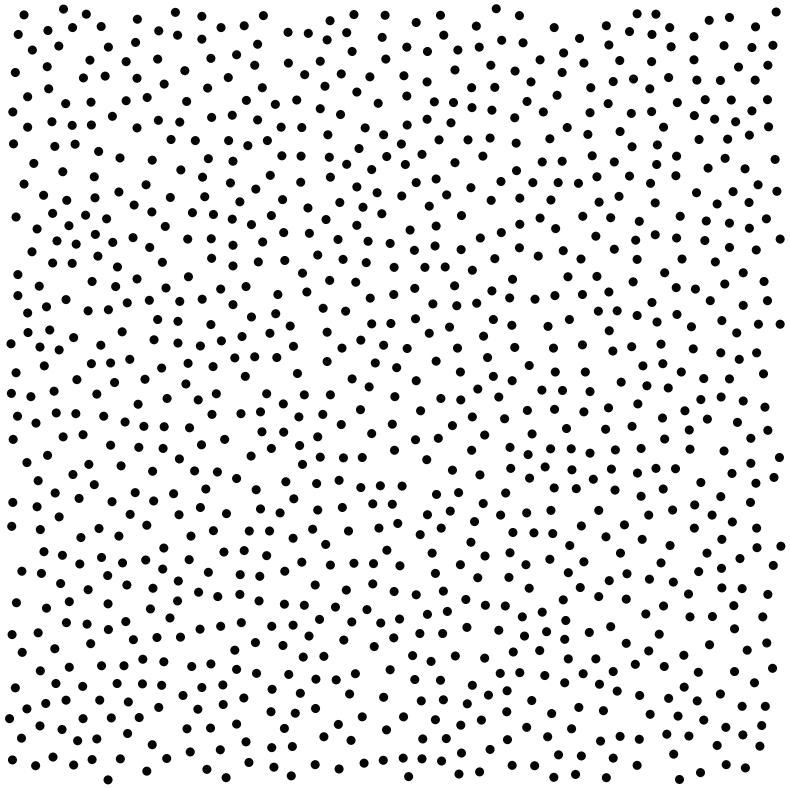

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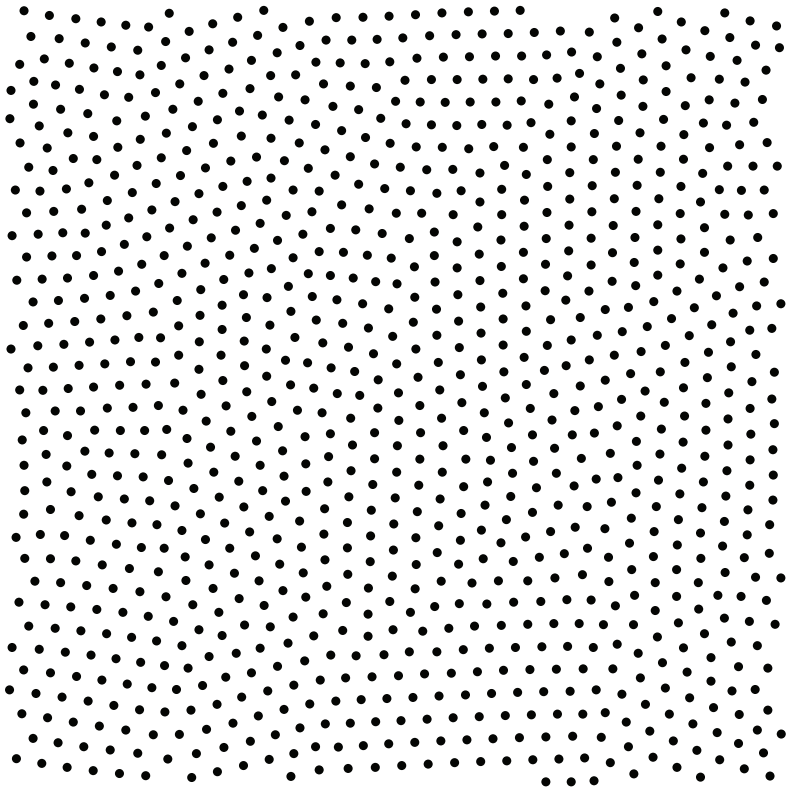

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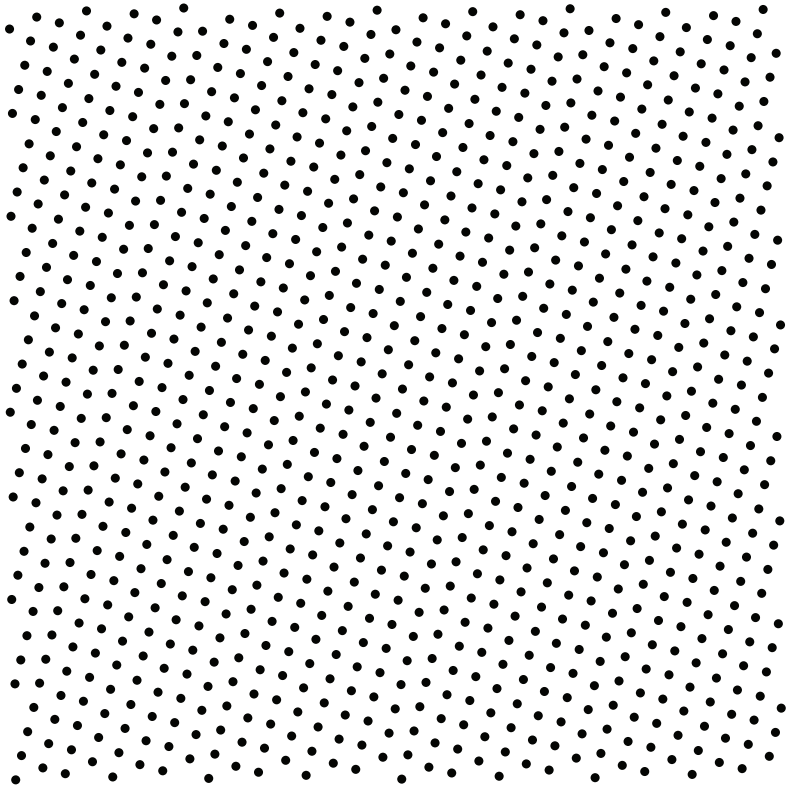

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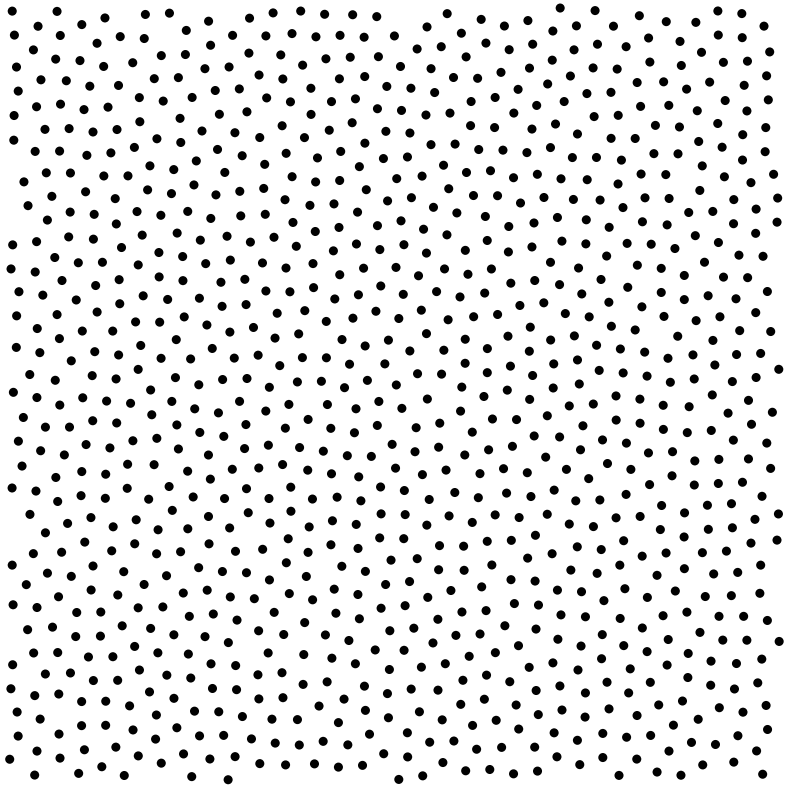

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Farthest-Point Optimization

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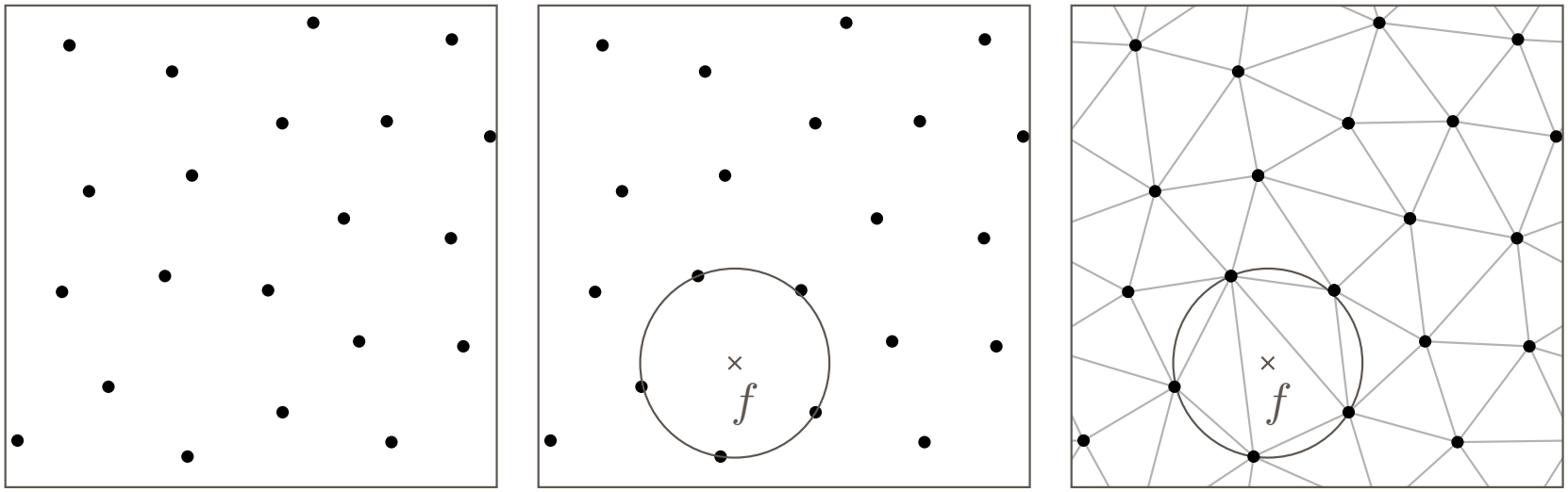
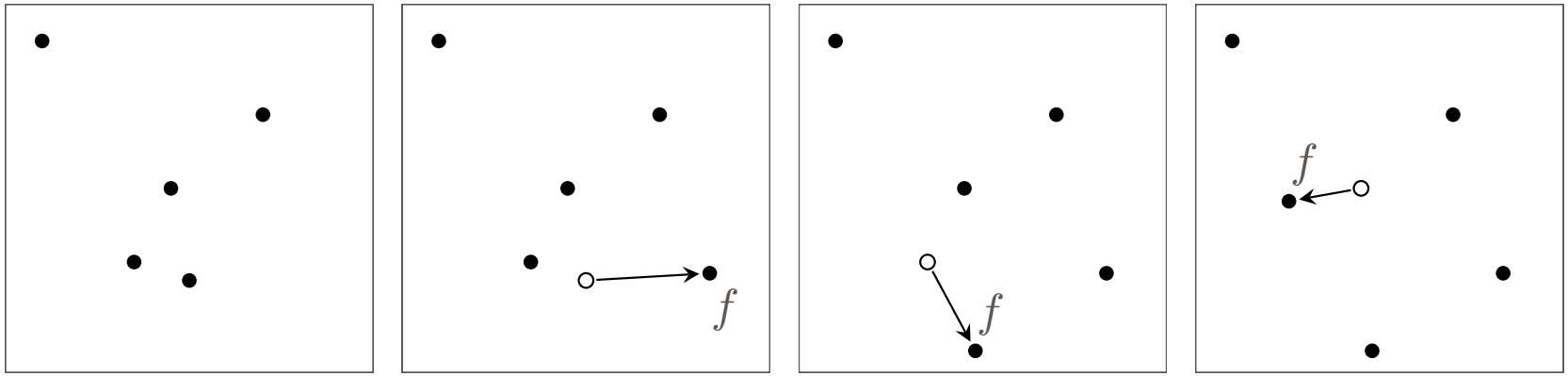
Farthest Point



- Location f that has the maximum distance from all points in X
- Corresponds to the center of the largest empty circle in the domain
- Corresponds to the center of the largest circumcircle of a triangle in the Delaunay triangulation $\mathcal{D}(X)$

Farthest-Point Optimization

Optimization Strategy



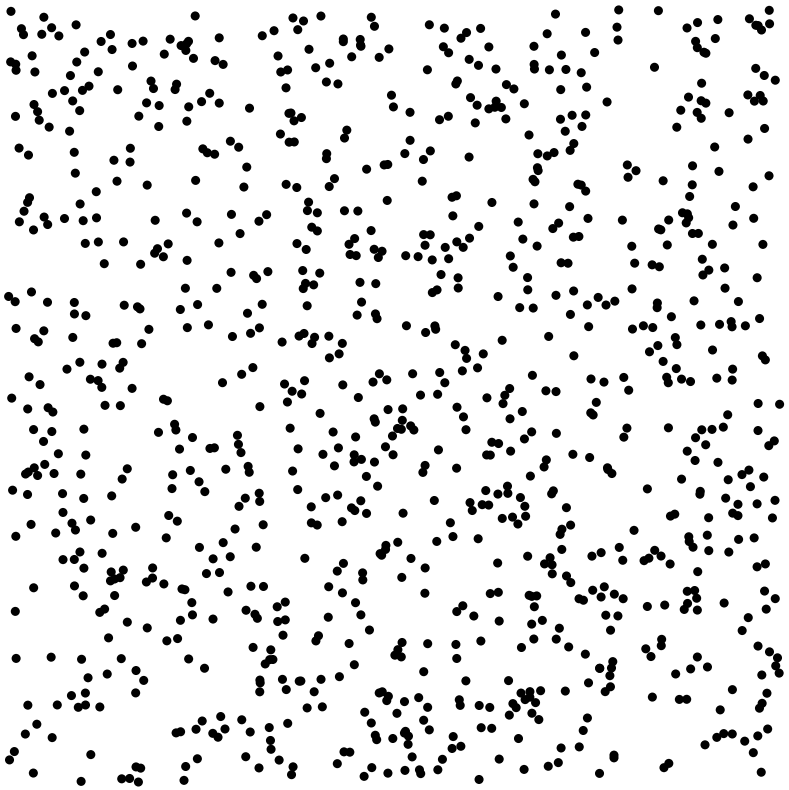
- Successively move each point to the farthest point, i.e. remove it and reinsert it at the farthest point
- One full iteration: move each point once
- Build full Delaunay triangulation once and update it dynamically

Main Algorithm

```
FARTHEST-POINT-OPTIMIZATION(X)
D = DELAUNAY(X)
repeat
    foreach vertex x in D
        (f, r_max) = (x, d_x)
        DELAUNAY-REMOVE(D, x)
        foreach triangle t in D
            (c, r) = center and radius of t's circumcircle
            if r > r_max
                (f, r_max) = (c, r)
        DELAUNAY-INSERT(D, f)
until converged
return vertices of D
```

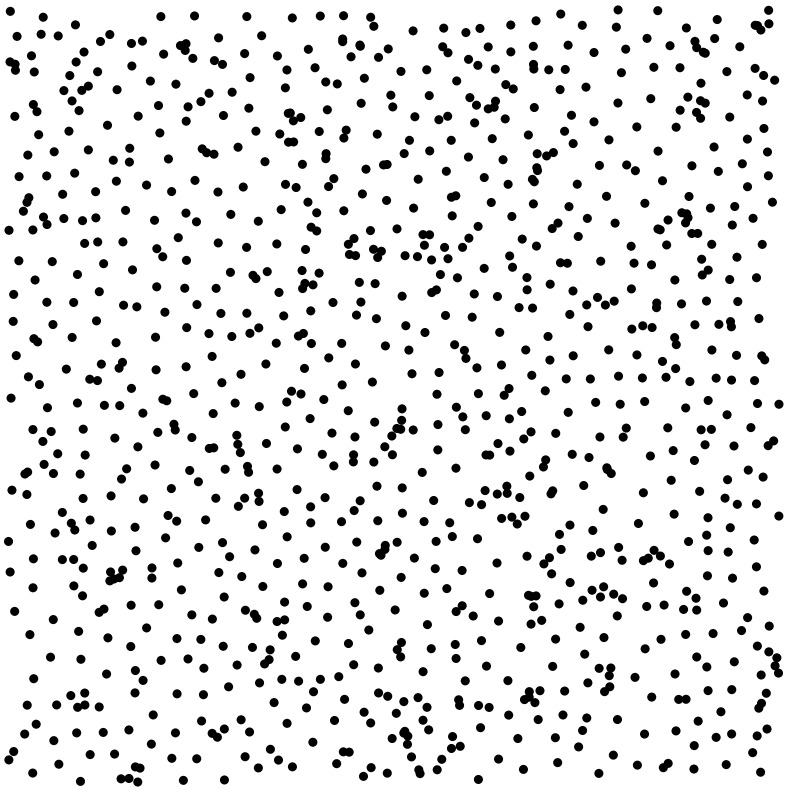
Farthest-Point Optimization

Optimizing Random Seed Points



$\delta_X \approx 0.009,\ \bar{\delta}_X \approx 0.469$

input

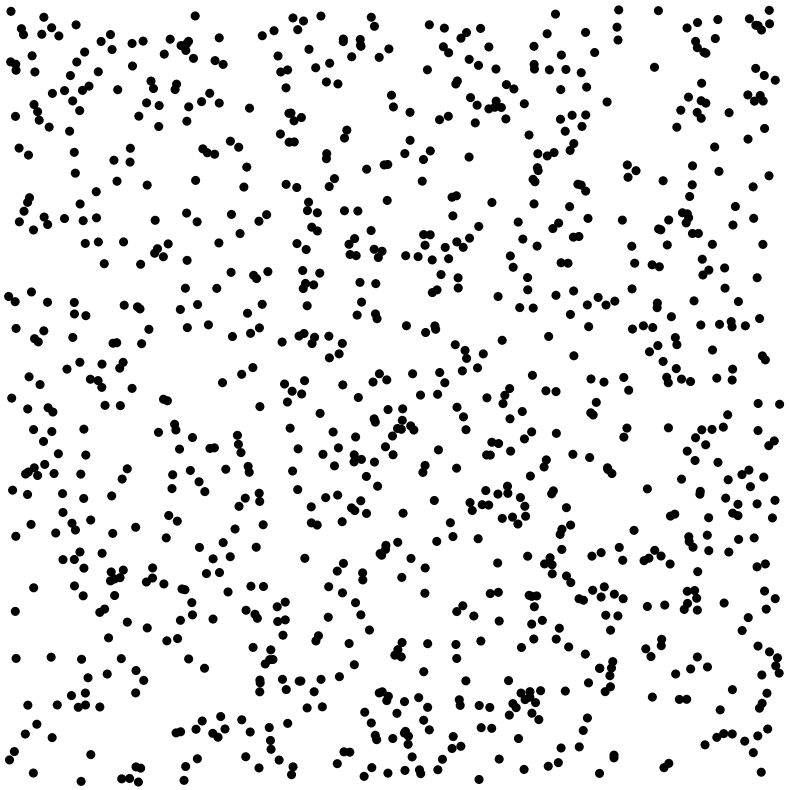


$\delta_X \approx 0.049,\ \bar{\delta}_X \approx 0.645$

1/4 iteration

Farthest-Point Optimization

Optimizing Random Seed Points



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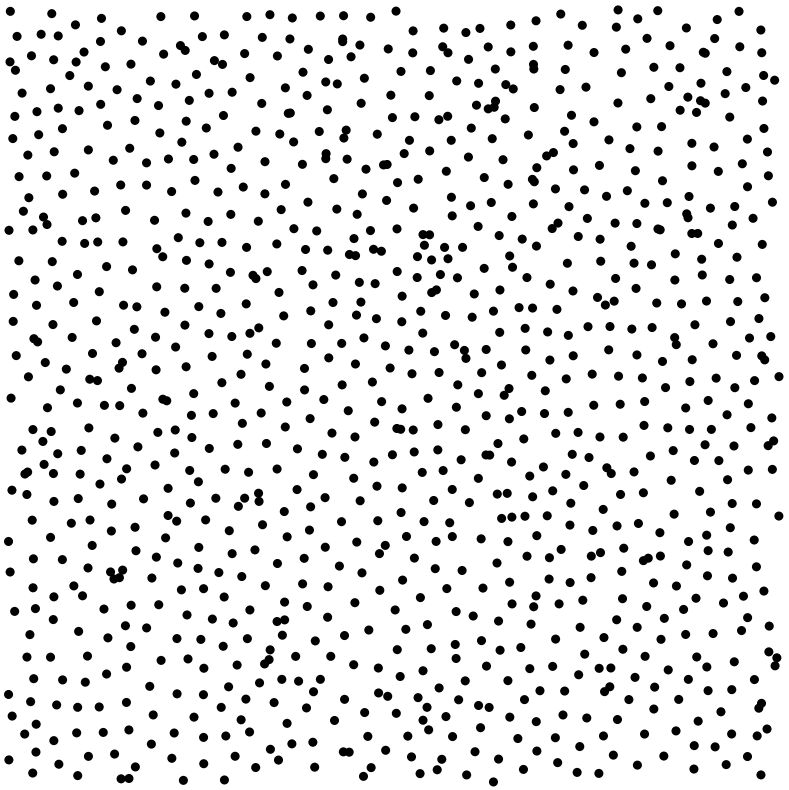

$\delta_X \approx 0.079,\ \bar{\delta}_X \approx 0.756$

1/2 iteration

Farthest-Point Optimization

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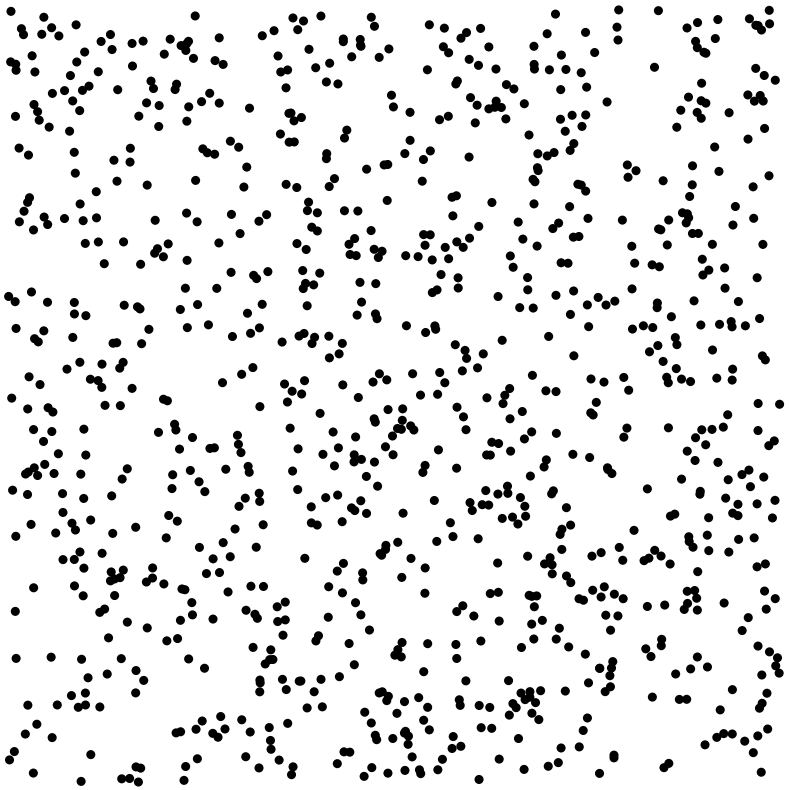

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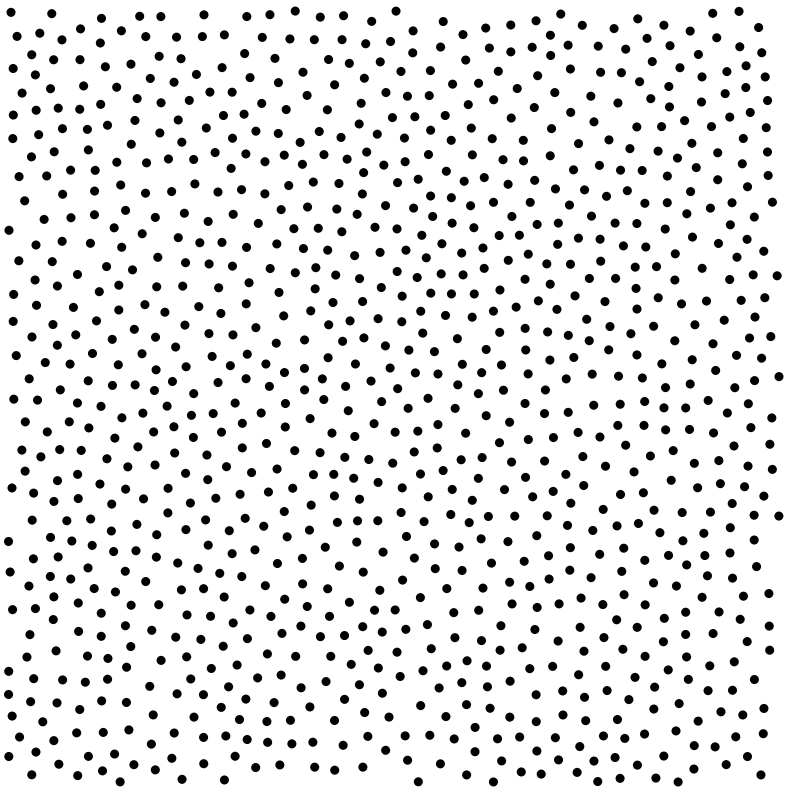
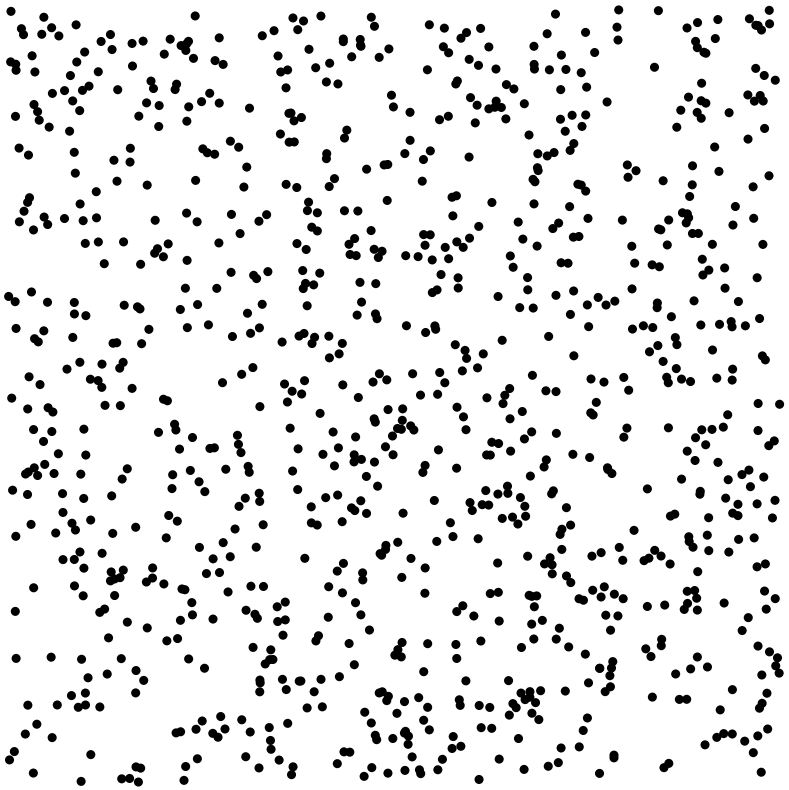


$\delta_X \approx 0.772,\ \bar{\delta}_X \approx 0.865$

1 iteration

Farthest-Point Optimization

Optimizing Random Seed Points



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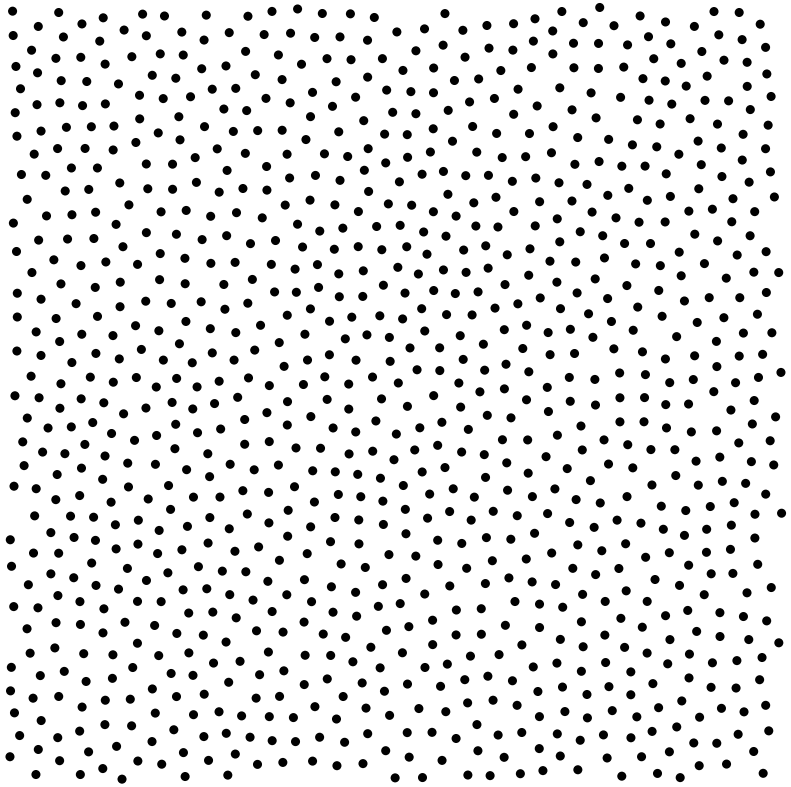
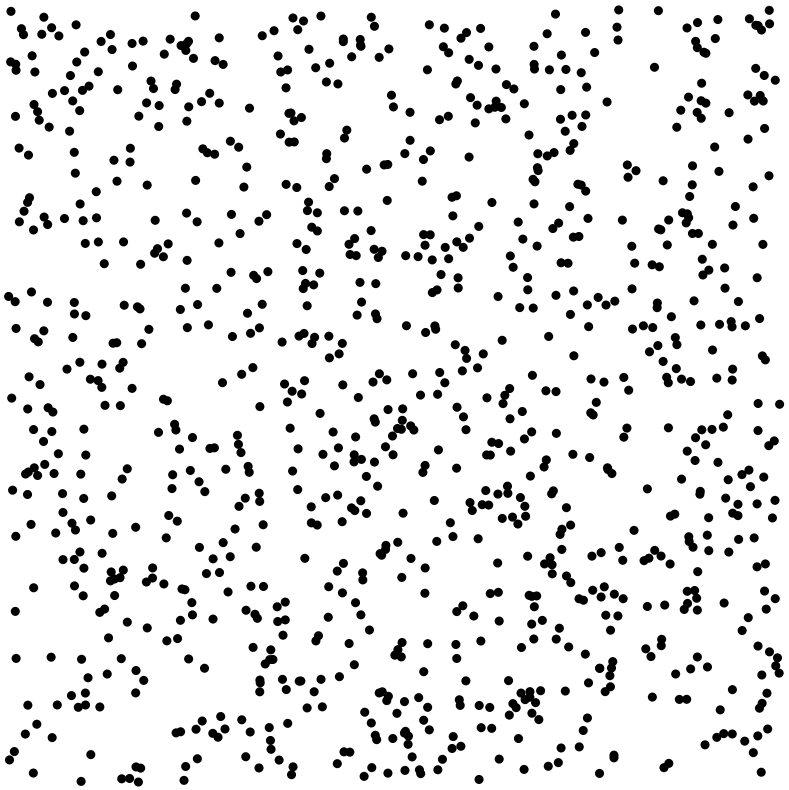


$\delta_X \approx 0.814,\ \bar{\delta}_X \approx 0.905$

2 iterations

Farthest-Point Optimization

Optimizing Random Seed Points



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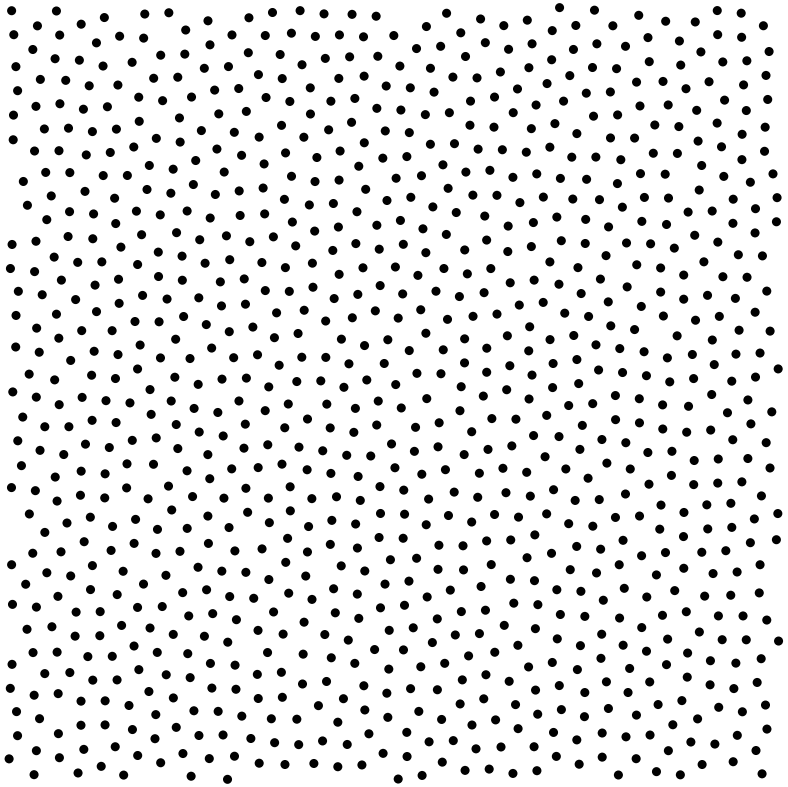
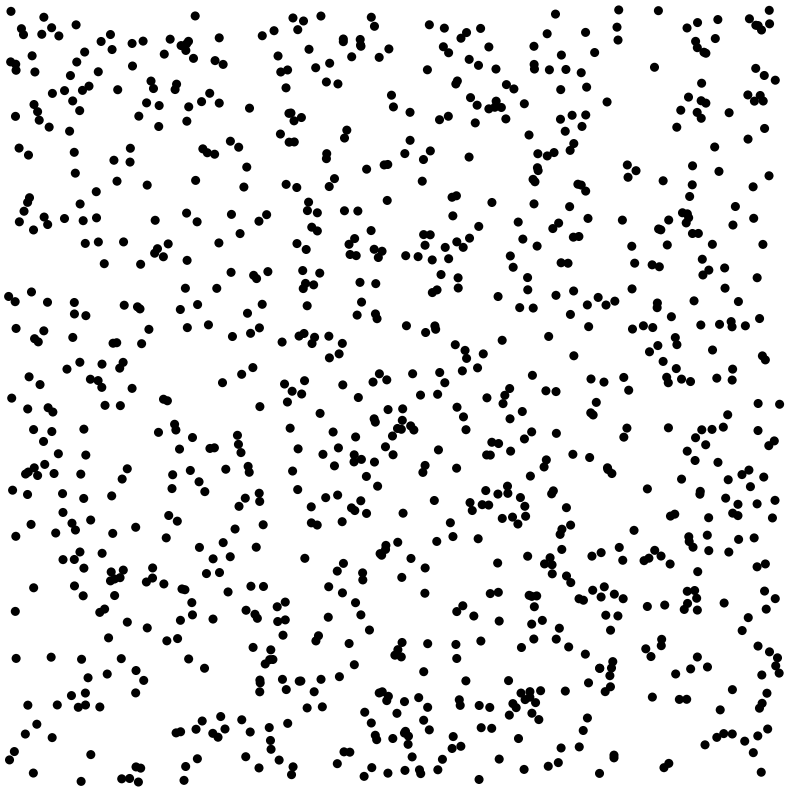


$\delta_X \approx 0.925,\ \bar{\delta}_X \approx 0.932$

63 iterations

Farthest-Point Optimization

Optimizing Random Seed Points



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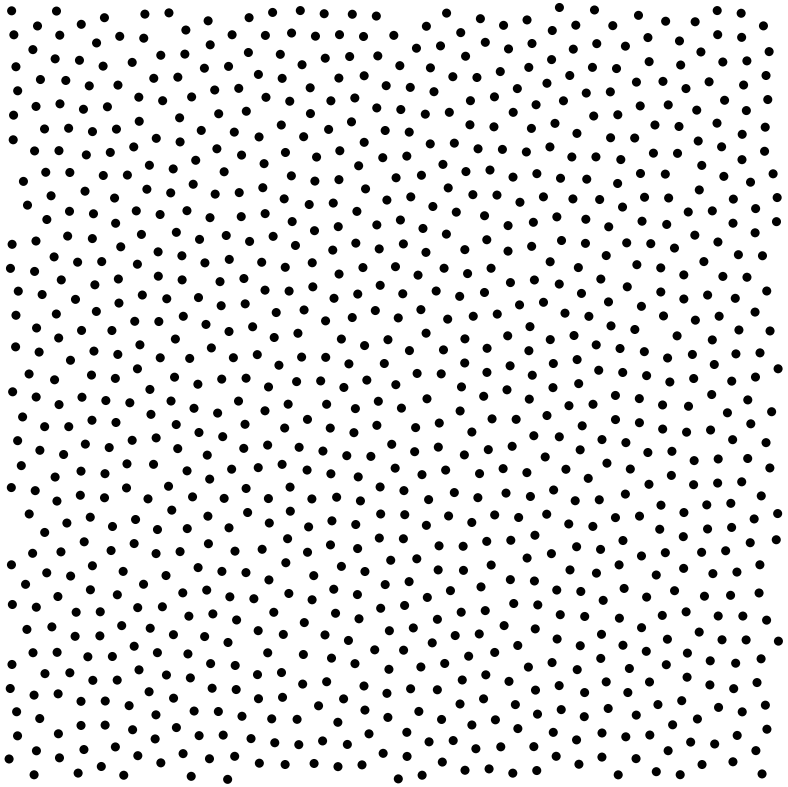

$\delta_X \approx 0.930,\ \bar{\delta}_X \approx 0.932$

419 iterations

Farthest-Point Optimization

Convergence

- Moving a point x maximizes by definition its local mindist δ_x
- Since $\bar{\delta}_X \propto \sum \delta_x \;\rightarrow\; \bar{\delta}_X$ increases strictly monotonically
- Stop when $\bar{\delta}_X^{\text{new}} - \bar{\delta}_X^{\text{old}} < \epsilon$ by machine precision

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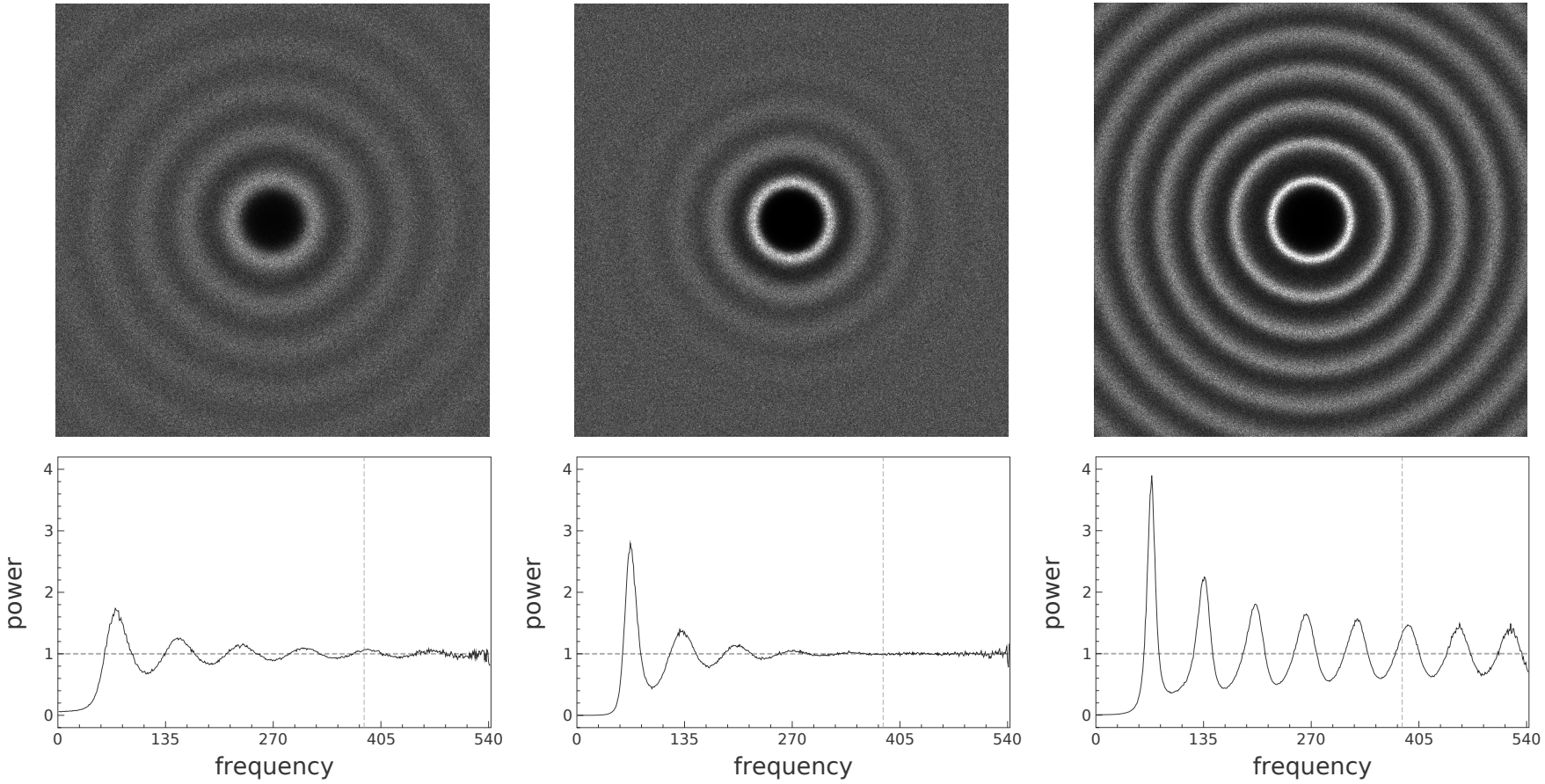
Runtime Complexity

- Dominated by the search operation for the largest circumcircle
- Utilizing a binary tree to track the largest circumcircle yields total complexity of $\mathcal{O}(n \log n)$ for a full iteration
- Delaunay operations in our case typically $\mathcal{O}(1)$ [Erickson 2005]
- In the paper: local variant with $\mathcal{O}(n)$ per iteration but slower convergence

Farthest-Point Optimization

Spectral Analysis



Dart Throwing | [Balzer et al. 2009] | *Farthest-Point Opt.*

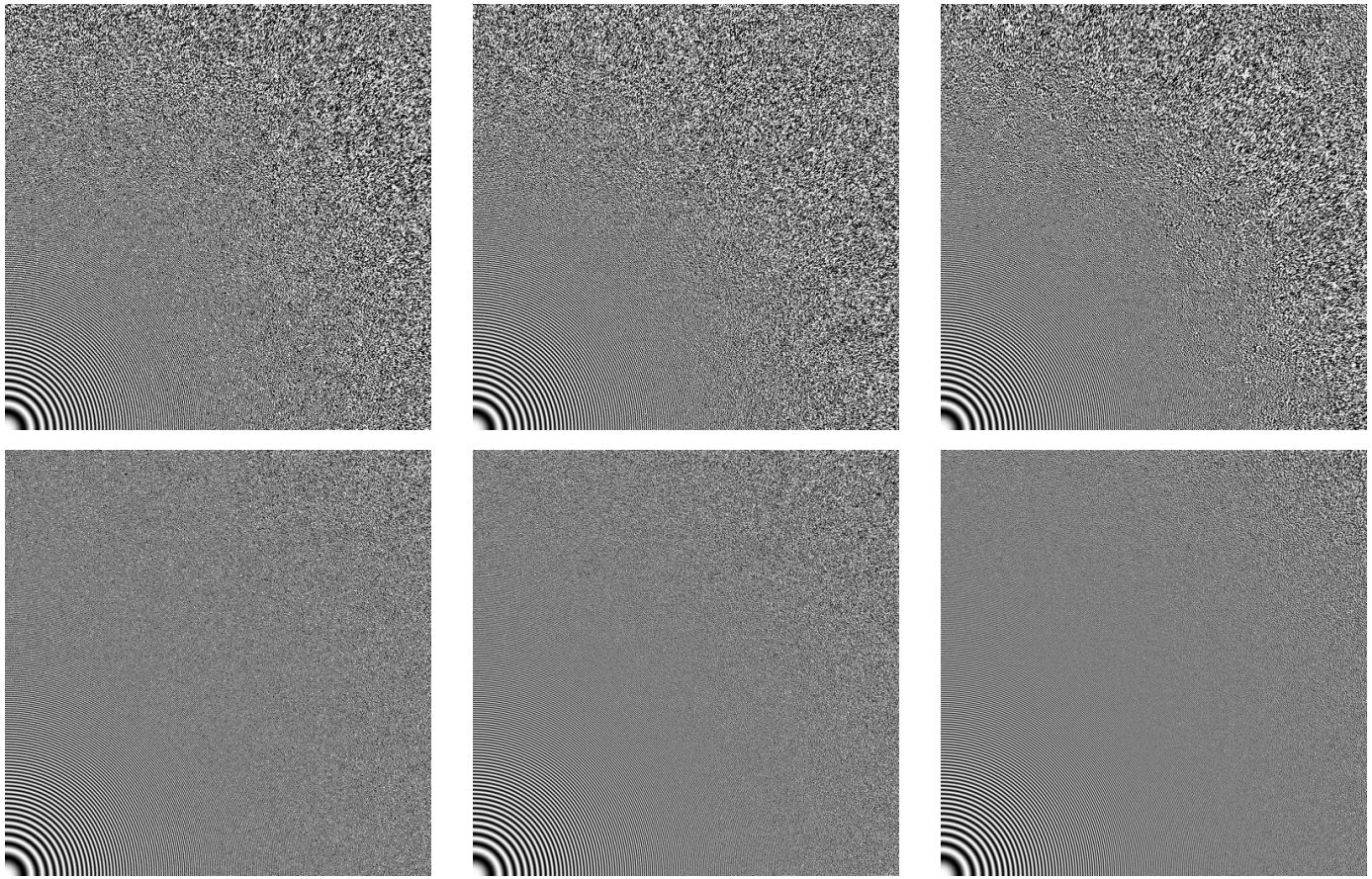
Application

Image Plane Sampling

- High mindist yields high effective Nyquist frequency

≈ 1spp

≈ 2spp



[Dunbar et al. 2006] [Schmaltz et al. 2010] *Farthest-Point Opt.*

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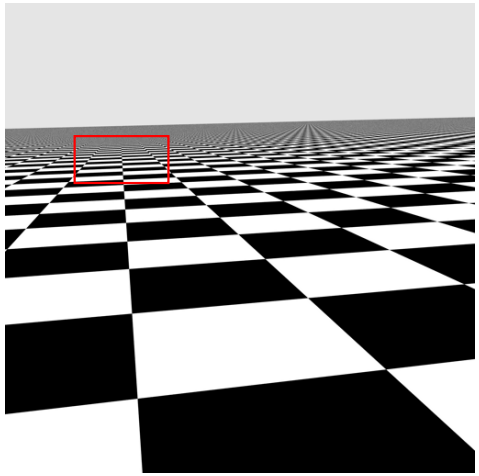
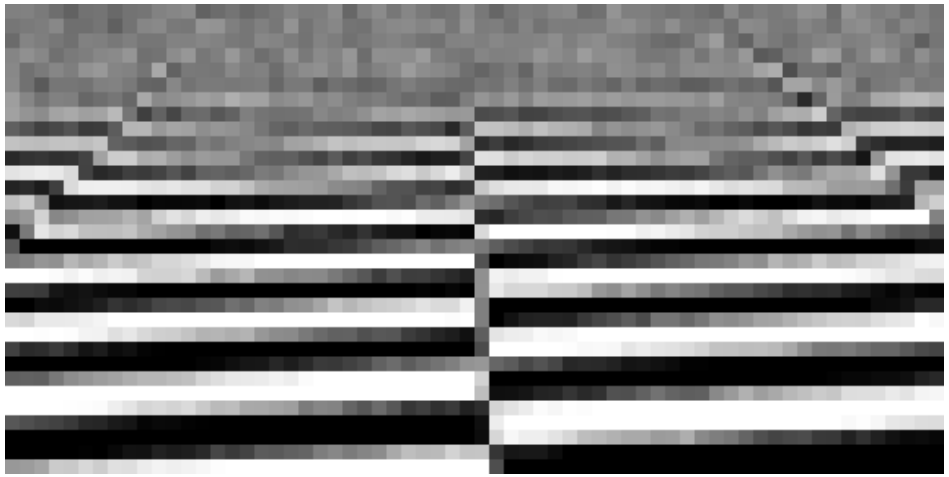


$\approx$ 16spp



$\mathrm{MSE} = 3.60 \cdot 10^{-3}$

Dart Throwing

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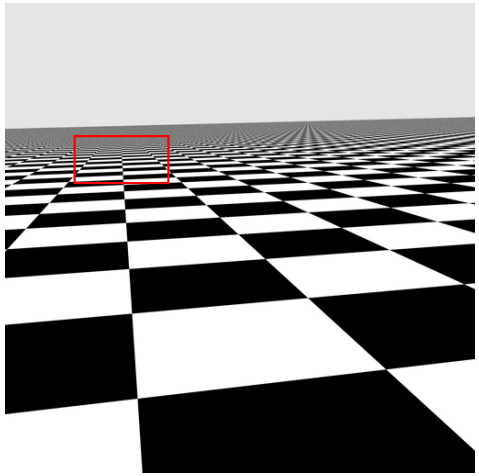

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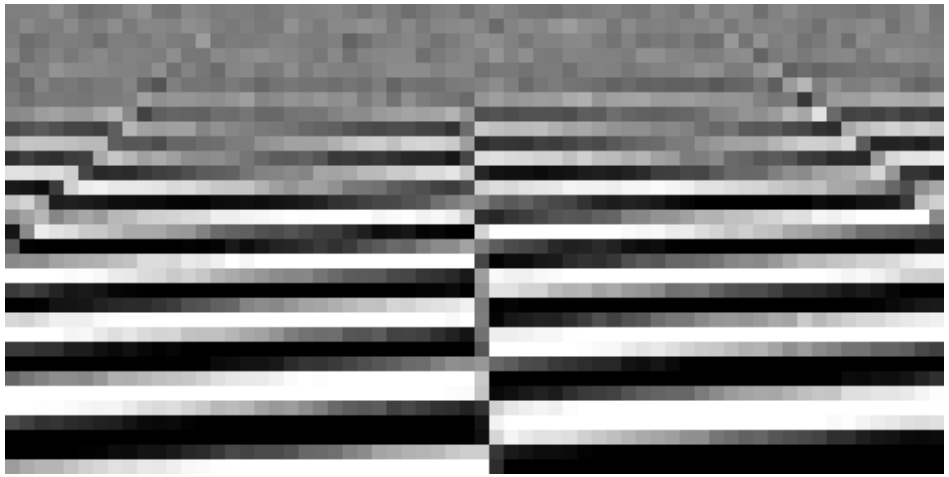


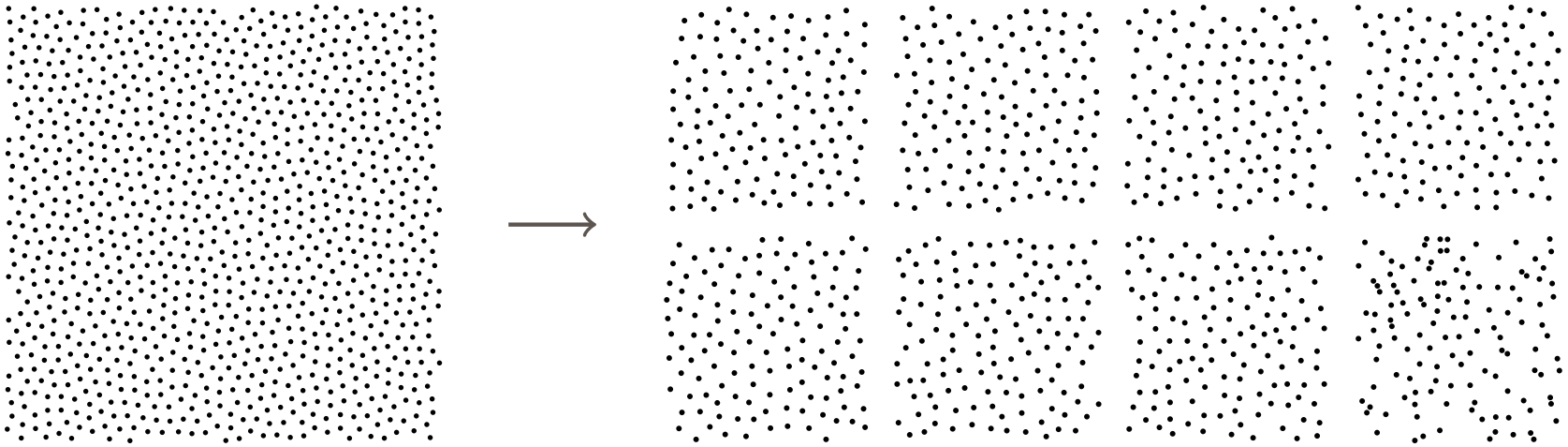
$\mathrm{MSE} = 2.58 \cdot 10^{-3}$

Farthest-Point Optimized

Extension

Extension

Partition Algorithm

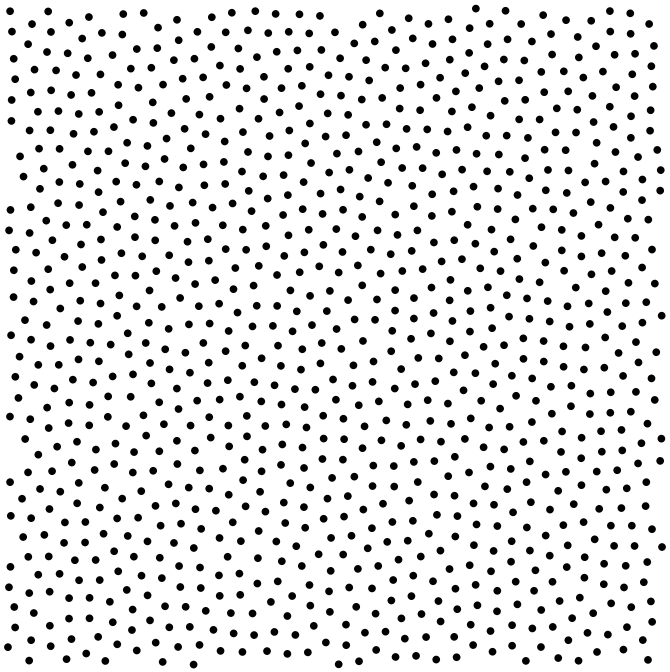


1024 points $\longrightarrow$ 8×128 points

- Want each subset as well as their union well-distributed
- Idea: partition input set into optimized subsets of equal size

Extension

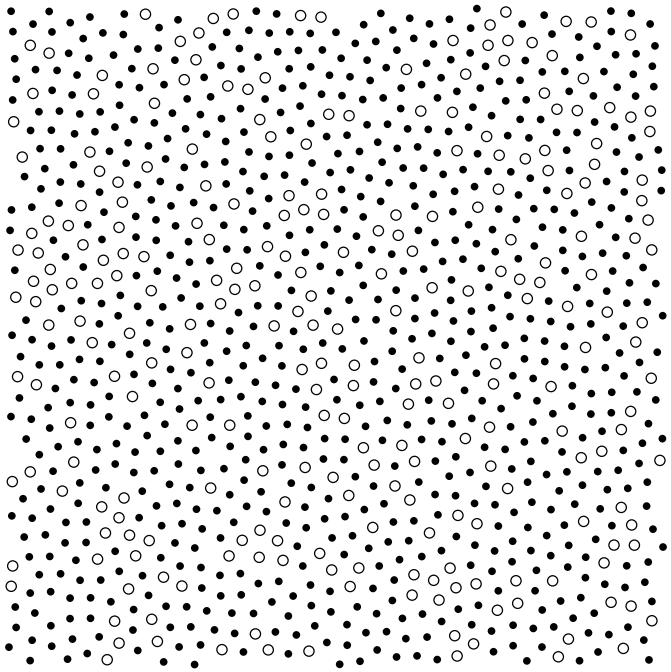
Partition Algorithm



- For each subset: discrete space variant of main FPO algorithm
- Main difference: f now bound to a point $f \in X$
- Greedy approach: construct subsets sequentially

Extension

Partition Algorithm



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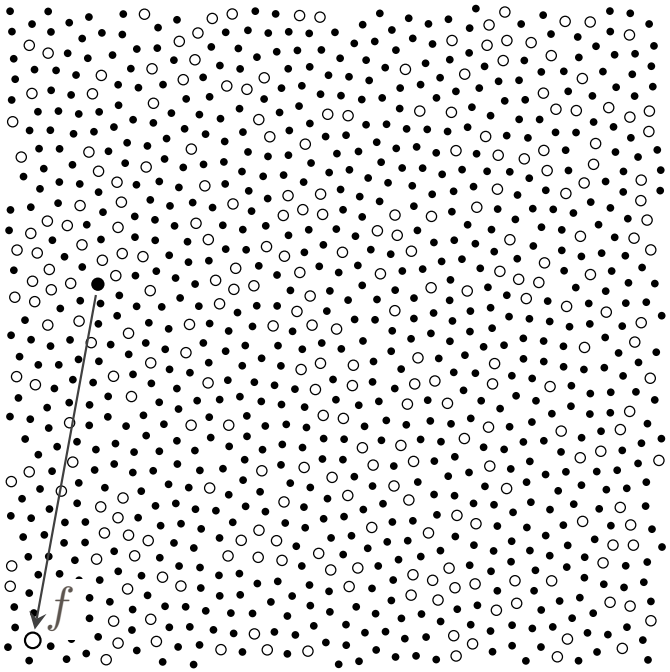

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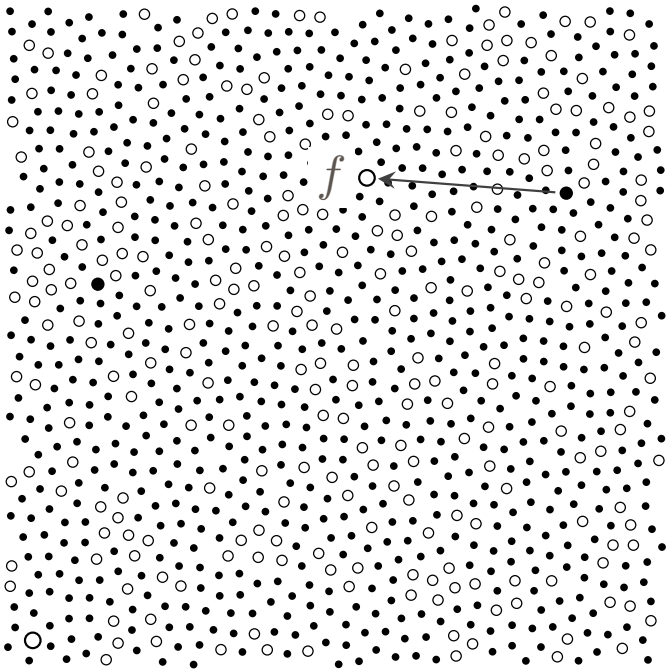

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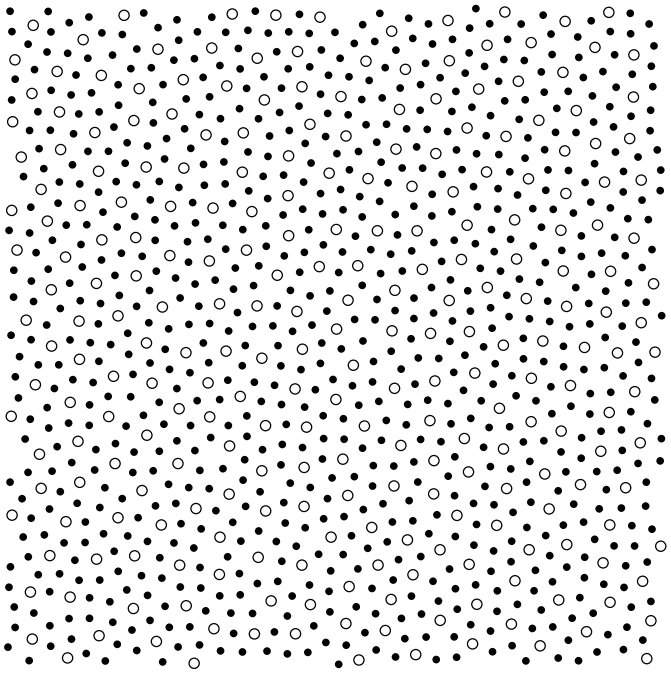

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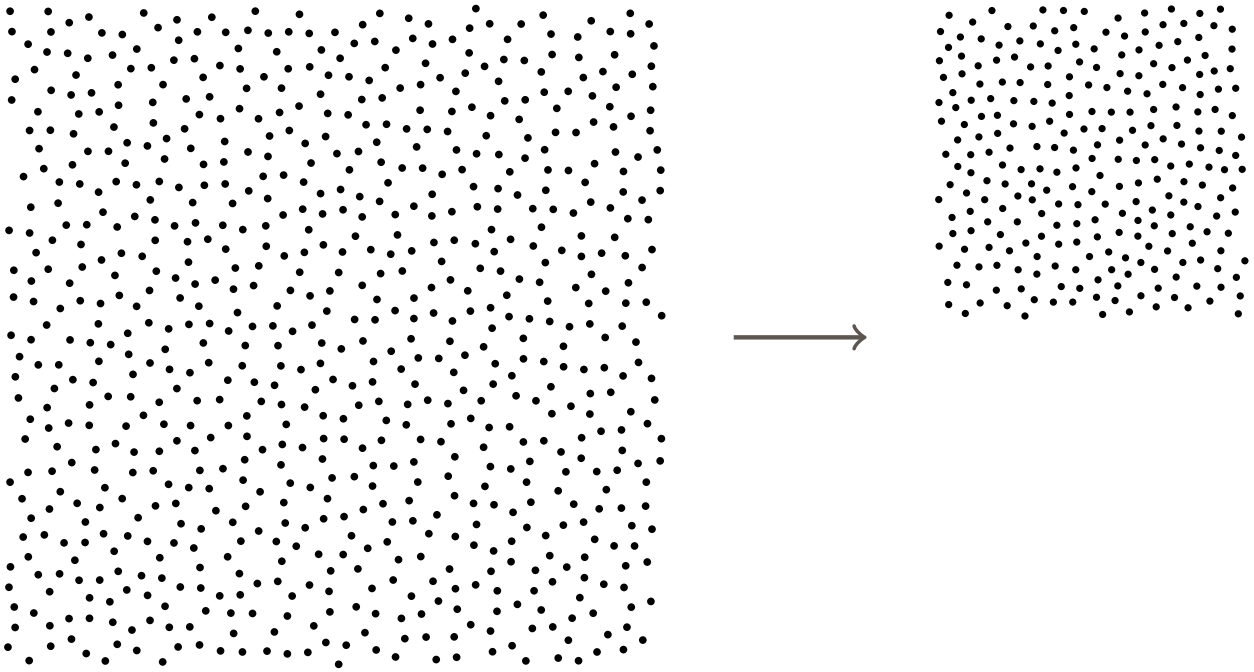

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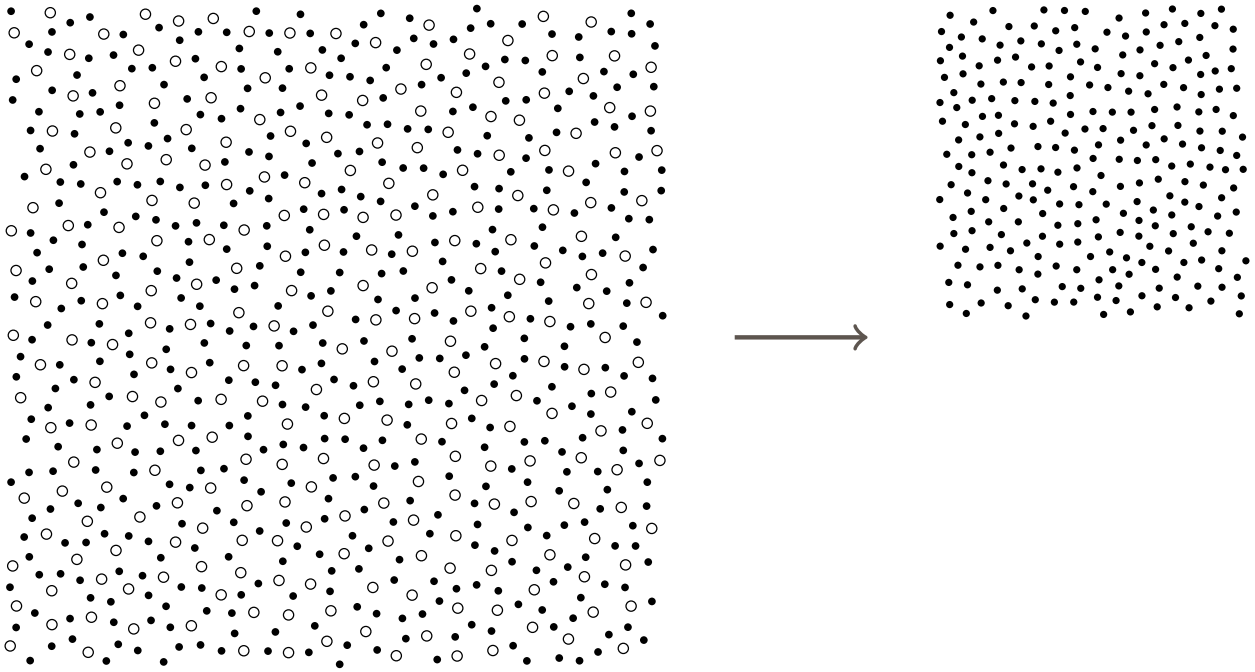

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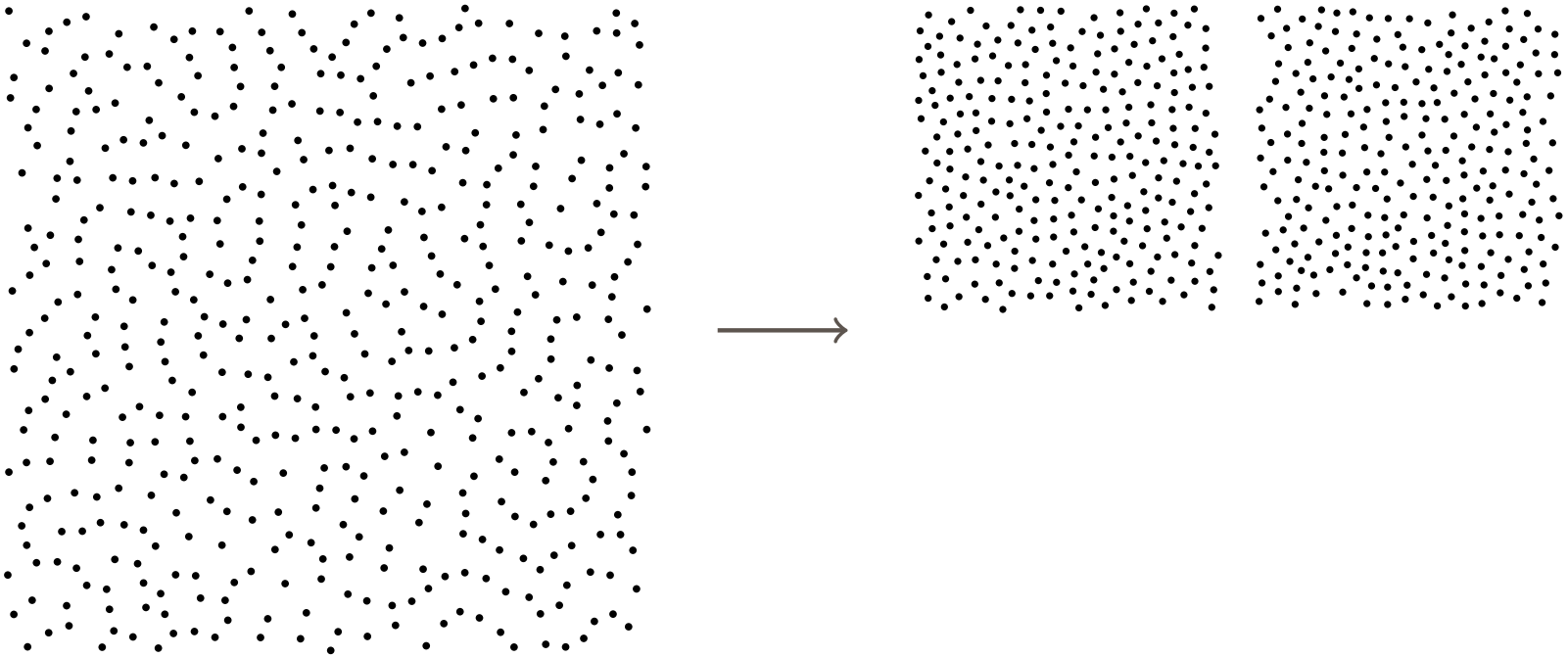

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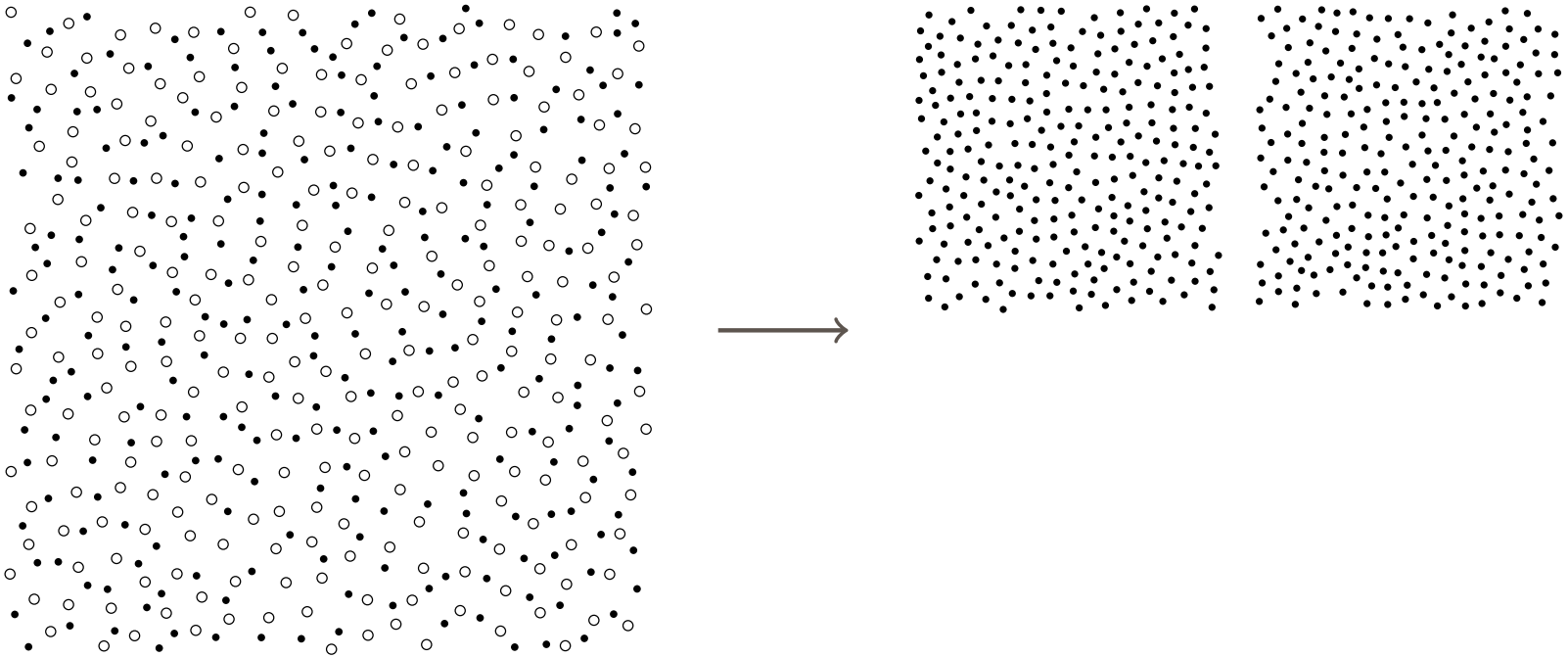

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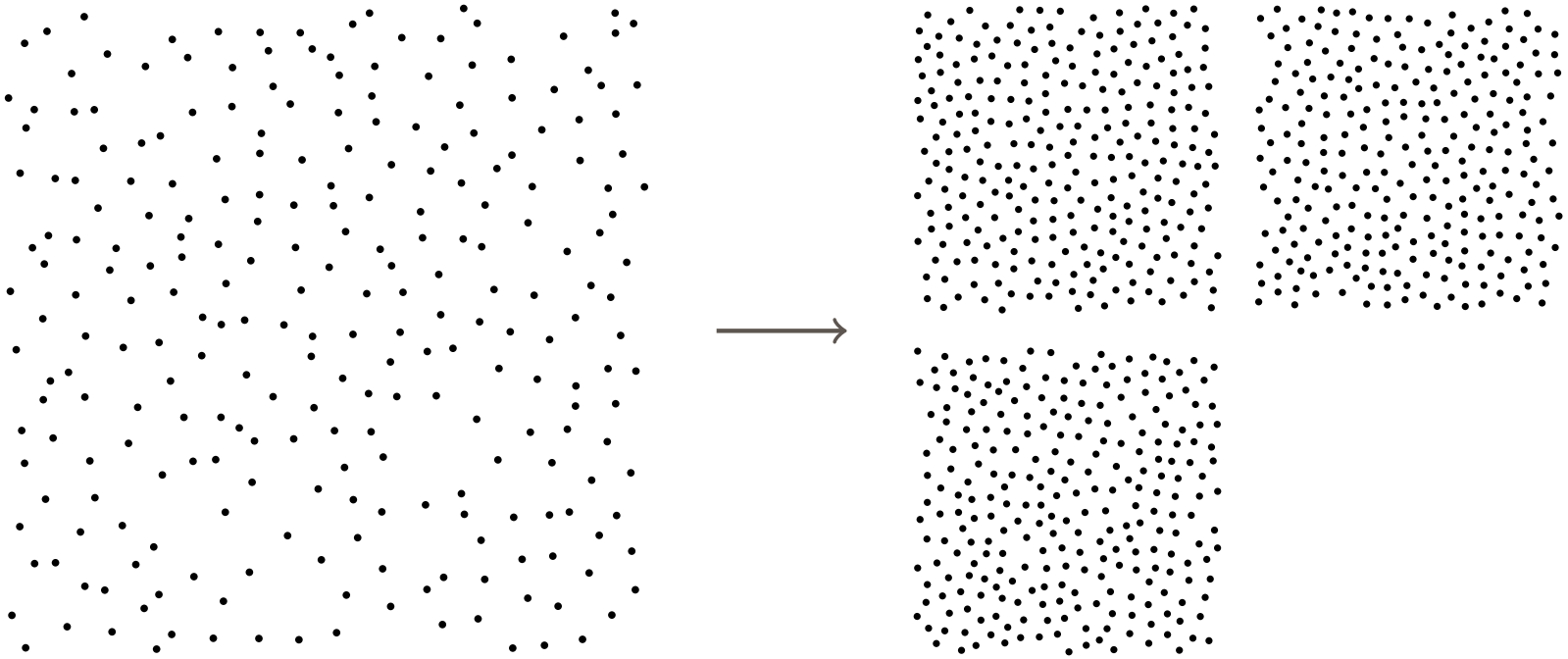

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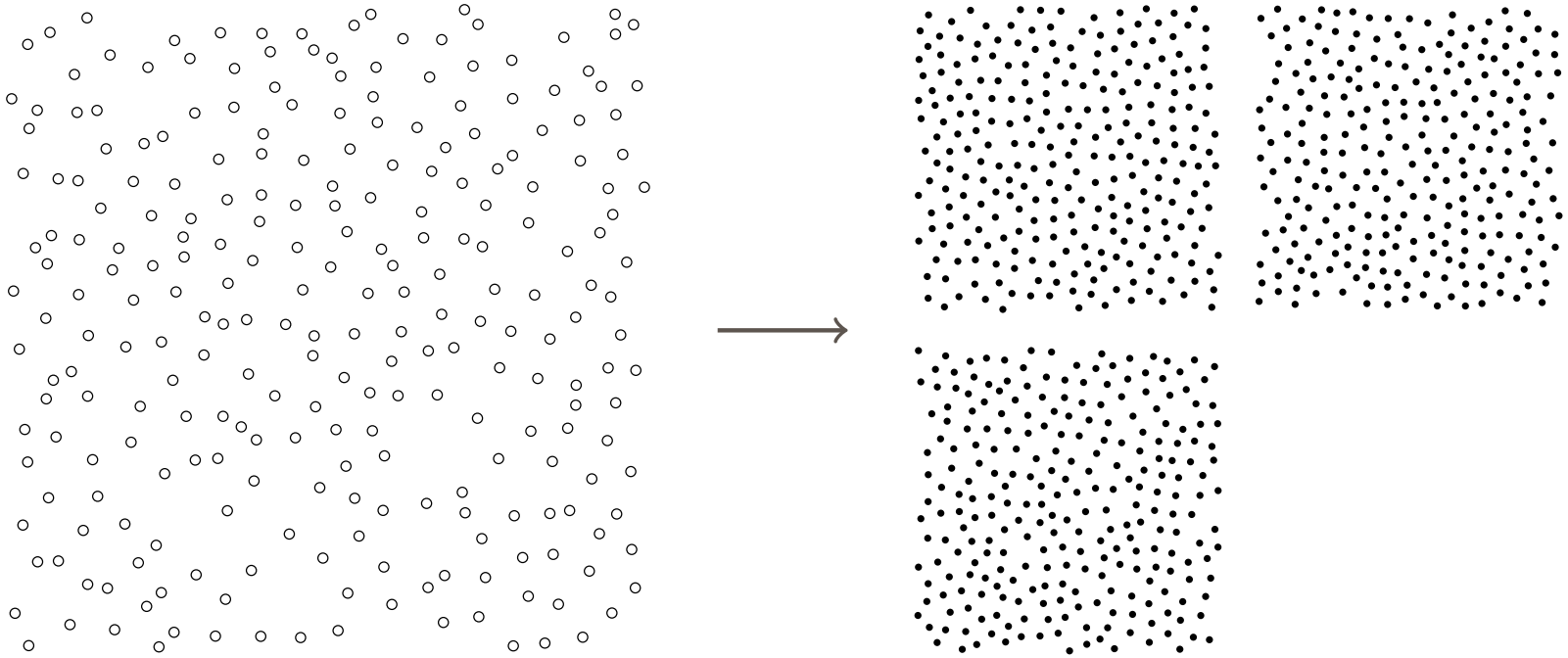

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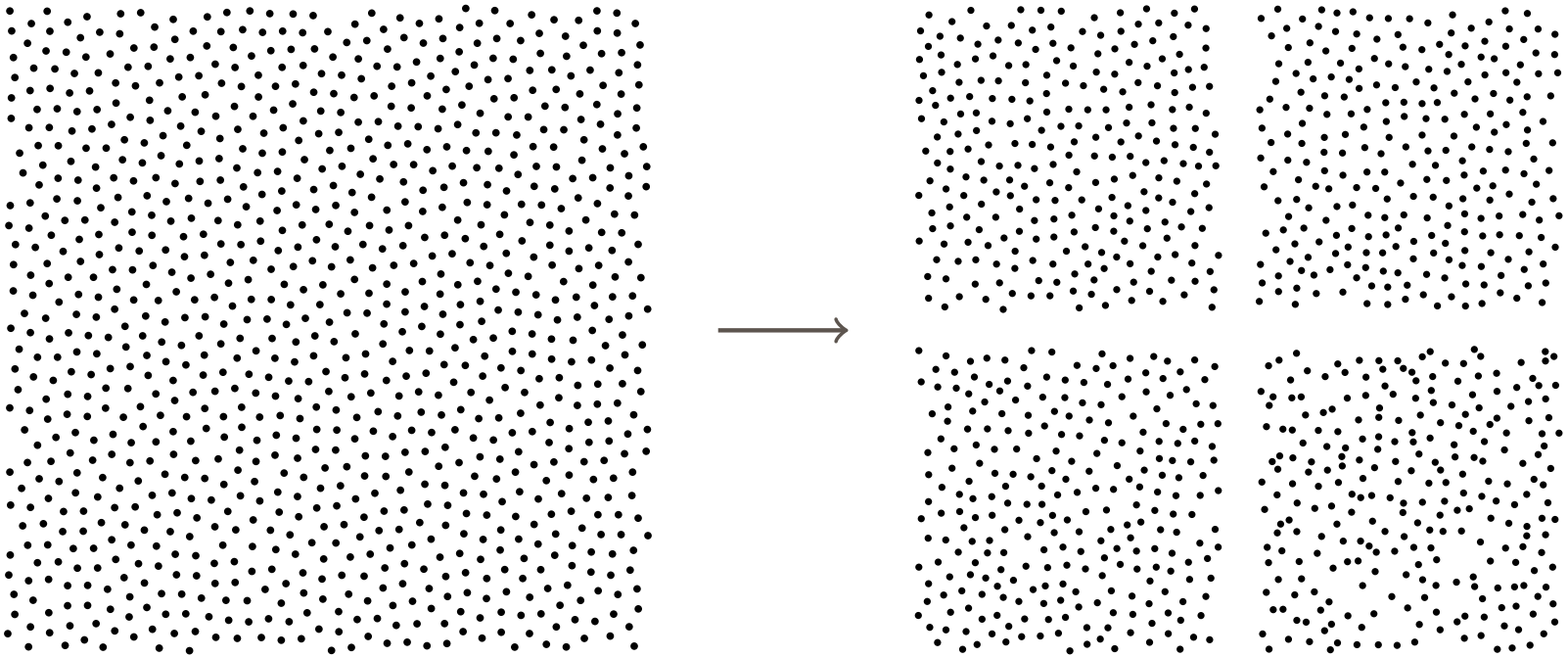

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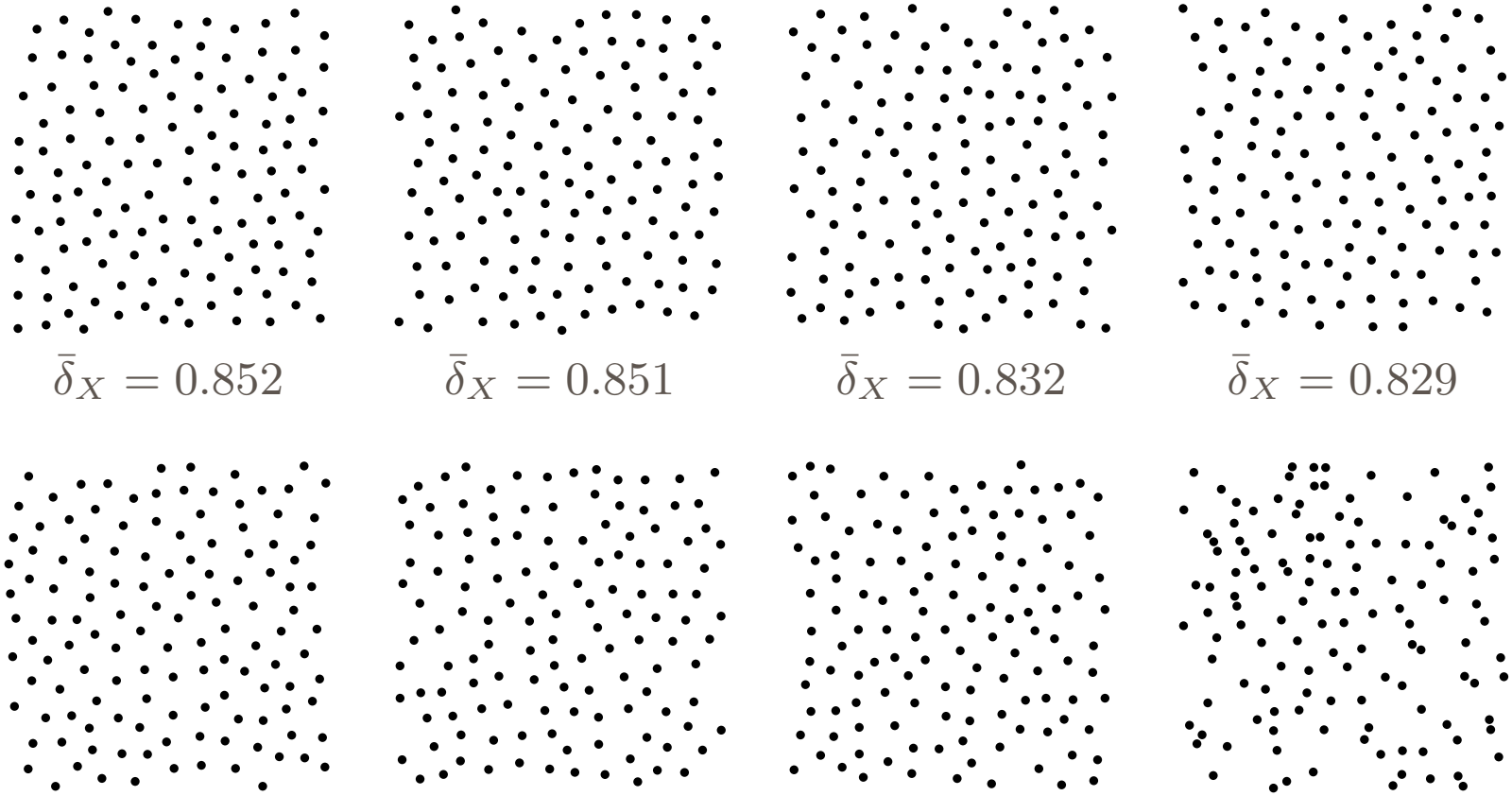
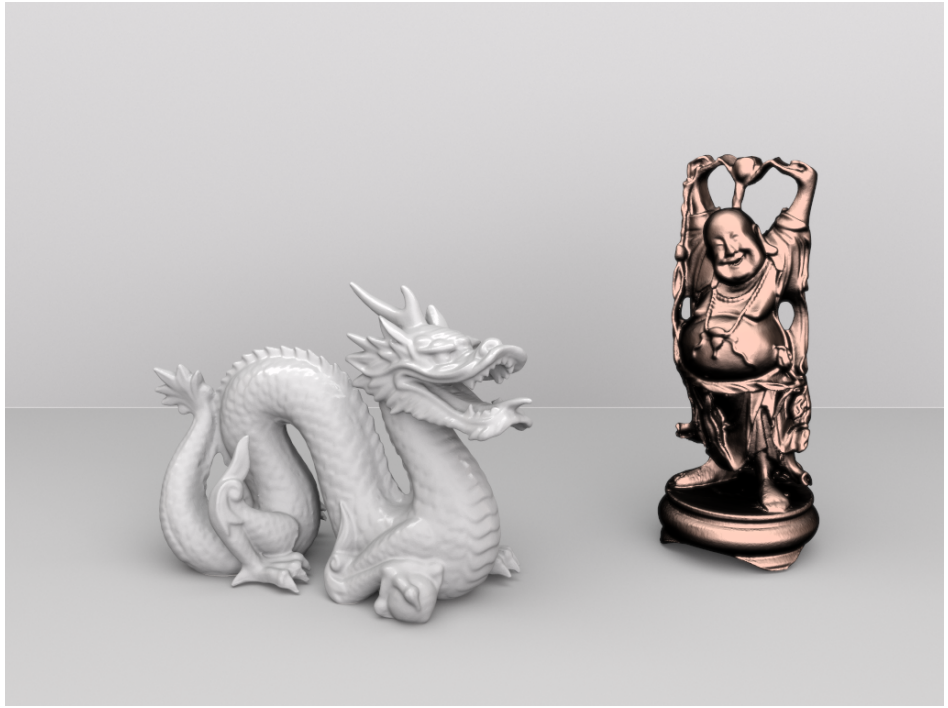


Application

Trajectory Splitting

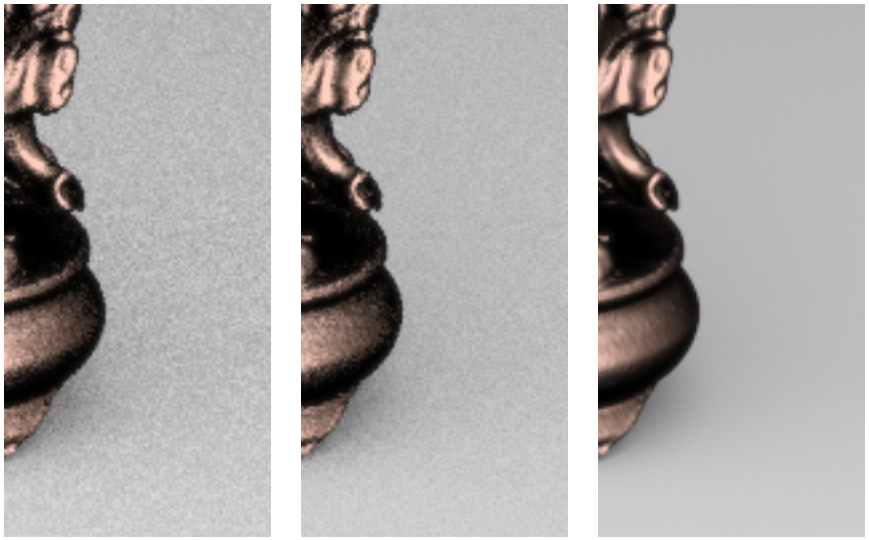


- Direct light estimation by trajectory splitting
- Precompute FPO points and necessary partitions
- Unbiased estimators by random shifts on unit torus

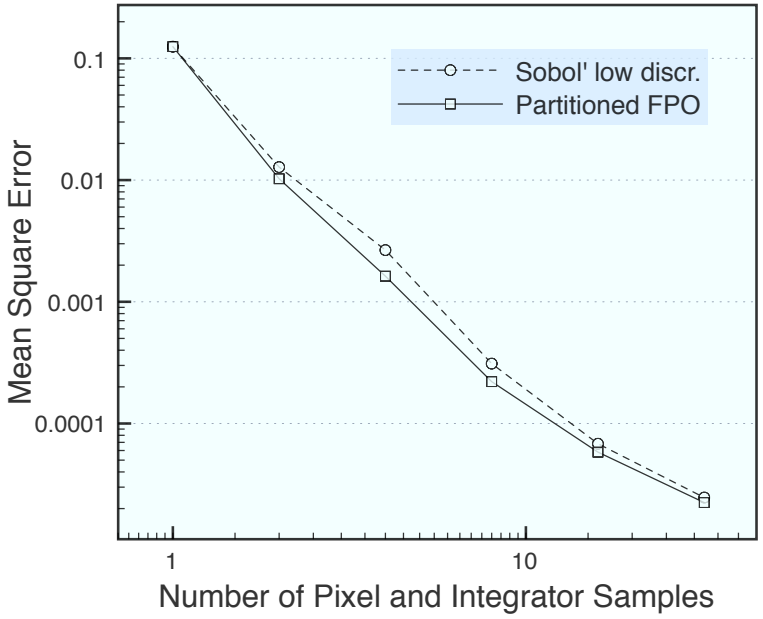
Application

Trajectory Splitting

4 × 4 samples



Sobol (0, 2) *FPO* reference



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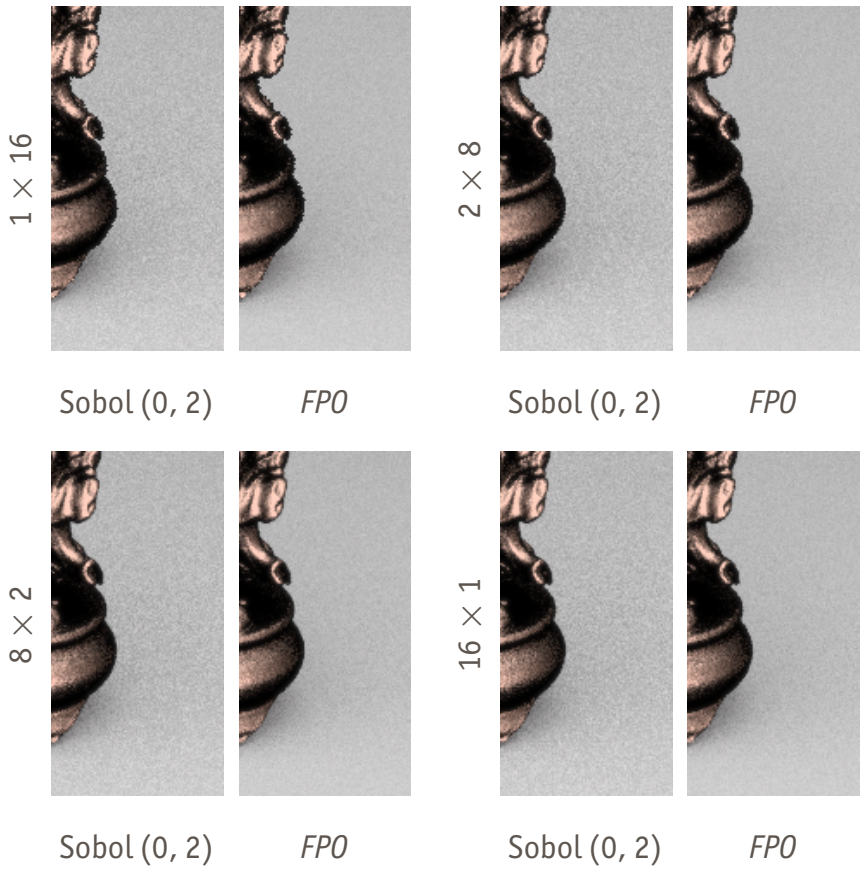

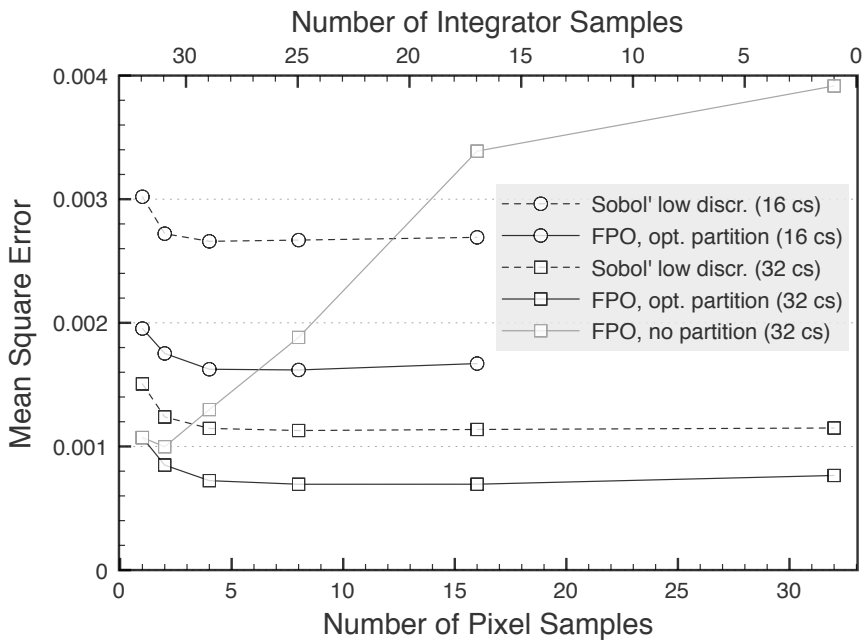

Conclusion

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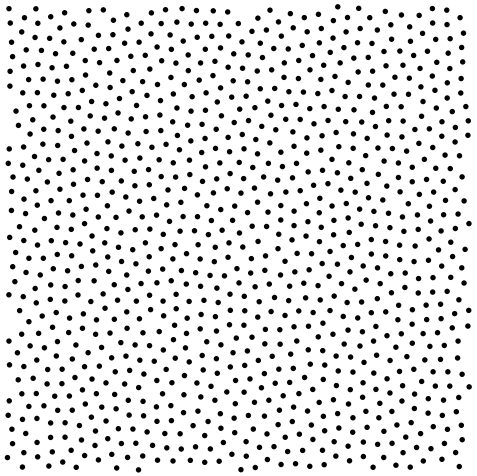
Farthest-Point Optimized Point Sets

- Uniform but irregular
- Higher mindist than previous methods
- Good trade-off between noise and coherent aliasing
- Optimal blue noise?



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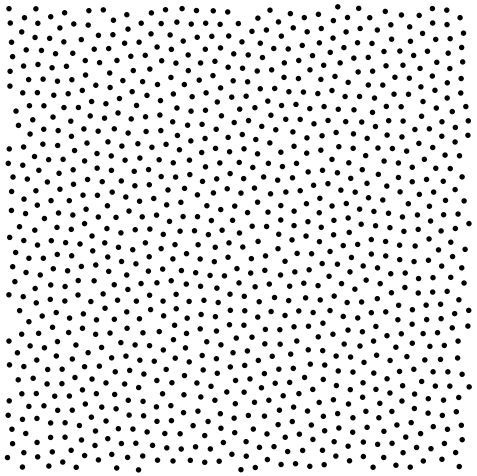

Partition Algorithm

- Good results but greedy approach leaves room for improvement

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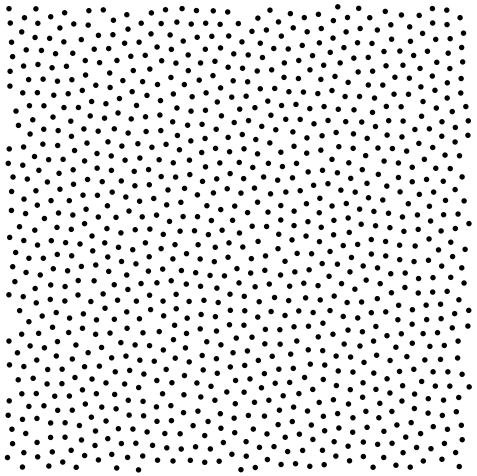

Partition Algorithm

- Good results but greedy approach leaves room for improvement

Future Work

- Points on bounded surfaces or triangulated domains
- Behavior in higher dimensions or for non-Euclidean metrics

Thank you for your attention.

References

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